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Extremal Problems for Nonvanishing Functions in Bergman Spaces

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Dedicated to the memory of Semeon Yakovlevich Khavinson.

Abstract. In this paper, we study general extremal problems for non-vanishing functions in Bergman spaces. We show the existence and uniqueness of solutions to a wide class of such problems. In addition, we prove certain regularity results: the extremal functions in the problems considered must be in a Hardy space, and in fact must be bounded. We conjecture what the exact form of the extremal function is. Finally, we discuss the specific problem of minimizing the norm of non-vanishing Bergman functions whose first two Taylor coefficients are given.

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1. Introduction

For $0 < p < \infty$, let

$$A^p = \{f \text{ analytic in } \mathbb{D} : (\int_{\mathbb{D}} |f(z)|^p dA(z))^{\frac{1}{p}} := \|f\|_{A^p} < \infty\}$$

denote the Bergman spaces of analytic functions in the unit disk $\mathbb{D}$. Here dA stands for normalized area measure $\frac{1}{\pi}dxdy$ in $\mathbb{D}$, $z = x + iy$. For $1 \le p < \infty$, A^p is a Banach space with norm $\| \; \|_{A^p}$. A^p spaces extend the well-studied scale of Hardy spaces

$$H^p := \{f \text{ analytic in } \mathbb{D} : (\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi})^{\frac{1}{p}} := \|f\|_{H^p} < \infty\}.$$

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For basic accounts of Hardy spaces, the reader should consult the well-known monographs [Du, Ga, Ho, Ko, Pr]. In recent years, tremendous progress has been achieved in the study of Bergman spaces following the footprints of the Hardy spaces theory. This progress is recorded in two recent monographs [HKZ, DS] on the subject.

In H^p spaces, the theory of general extremal problems has achieved a state of finesse and elegance since the seminal works of S.Ya. Khavinson, and Rogosinski and Shapiro (see [Kh1, RS]) introduced methods of functional analysis. A more or less current account of the state of the theory is contained in the monograph [Kh2]. However, the theory of extremal problems in Bergman spaces is still at a very beginning. The main difficulty lies in the fact that the Hahn-Banach duality that worked such magic for Hardy spaces faces tremendous technical difficulty in the context of Bergman spaces because of the subtlety of the annihilator of the A^p space ($p \geq 1$) inside $L^p(dA)$. [KS] contains the first more or less systematic study of general linear extremal problems based on duality and powerful methods from the theory of nonlinear degenerate elliptic PDEs. One has to acknowledge, however, the pioneering work of V. Ryabych [Ry1, Ry2] in the 60s in which the first regularity results for solutions of extremal problems were obtained. Vukotić's survey ([Vu]) is a nice introduction to the basics of linear extremal problems in Bergman space. In [KS], the authors considered the problem of finding, for $1 < p < \infty$,

$$\sup\{|\int_{\mathbb{D}} \bar{w} f dA| : \|f\|_{A^p} \leq 1\}, \tag{1.1}$$

where w is a given rational function with poles outside of $\mathbb{D}$. They obtained a structural formula for the solution (which is easily seen to be unique) similar to that of the Hardy space counterpart of problem (1.1). Note here that by more or less standard functional analysis, problem (1.1) is equivalent to

$$\inf\{\|f\|_{A^p} : f \in A^p, l_i(f) = c_i, i = 1, \ldots, n\}, \tag{1.2}$$

where the $l_i \in (A^p)^*$ are given bounded linear functionals on A^p, $p > 1$. Normally, for l_i one takes point evaluations at fixed points of $\mathbb{D}$, evaluations of derivatives, etc... More details on the general relationship between problems (1.1) and (1.2) can be found in [Kh2, pp. 69-74]. For a related discussion in the Bergman spaces context, we refer to [KS, p. 960]. In this paper, we focus our study on problem (1.2) for nonvanishing functions. The latter condition makes the problem highly nonlinear and, accordingly, the duality approach does not work. Yet, in the Hardy spaces context, in view of the parametric representation of functions via their boundary values, one has the advantage of reducing the nonlinear problem for nonvanishing functions to the linear problem for their logarithms. This allows one to obtain the general structural formulas for the solutions to problems (1.1) or (1.2) for nonvanishing functions in Hardy spaces as well. We refer the reader to the corresponding sections in [Kh2] and the references cited there. Also, some of the specific simpler problems for nonvanishing H^p functions have recently been solved in [BK]. However, all the above-mentioned methods fail miserably in the context of Bergman spaces for the simple reason that there are no non-trivial Bergman functions that, acting as multiplication operators on Bergman spaces, are isometric.

Let us briefly discuss the contents of the paper. In Section 2, we study problem (1.2) for nonvanishing Bergman functions: we show the existence and uniqueness of the solutions to a wide class of such problems. Our main results are presented in Sections 2 and 3 and concern the regularity of the solutions: we show that although posed initially in A^p, the solution must belong to the Hardy space H^p, and hence, as in the corresponding problems in Hardy spaces in [Kh2], must be a product of an outer function and a singular inner function. Further, we show that that the solutions to such problems are in fact bounded. Moreover, led by an analogy with the Hardy space case, we conjecture that the extremal functions have the form

$$f^*(z) = \exp\Big(\sum_{j=1}^{k} \lambda_j \frac{e^{i\theta_j}+z}{e^{i\theta_j}-z}\Big) \prod_{j=1}^{2n-2} (1-\bar{\alpha_j} z)^{\frac{2}{p}} \prod_{j=1}^{n} (1-\bar{\beta}_j z)^{-\frac{4}{p}}, \tag{1.3}$$

where $|\alpha_j| \leq 1$, $|\beta_j| < 1$, $\lambda_j < 0$, $n \geq 1$, $k \leq 2n-2$. In Section 4, we sketch how, if one knew some additional regularity of the solutions, it would be possible to derive the form (1.3) for the solutions. In the context of linear problems, i.e., with the nonvanishing restriction removed, duality can be applied and then, incorporating PDE machinery to establish the regularity of the solutions to the dual problem, the structural formulas for the solutions of (1.1) and (1.2) are obtained (see [KS]). We must stress again that due to the nonlinear nature of extremal problems for nonvanishing functions, new techniques are needed to establish the regularity of solutions up to the boundary beyond membership in an appropriate Hardy class. In the last section, we discuss a specific case of Problem 1.2 with $l_1(f) = f(0)$ and $l_2(f) = f'(0)$. The study of this simple problem was initiated by D. Aharonov and H.S. Shapiro in unpublished reports [AhSh1, AhSh2], and B. Korenblum has drawn attention to this question on numerous occasions.

2. Existence and regularity of solutions

Consider the following general problem.

Problem 2.1. *Given n continuous linearly independent linear functionals $l_1, l_2, \ldots,$ l_n on A^p and given n points $c_1, c_2, \ldots, c_n$ in $\mathbb{C} - \{0\}$, find*

$$\lambda = \inf\{\|f\|_{A^p} : f \text{ is zero-free, } l_i(f) = c_i, 1 \leq i \leq n\}.$$

The set of zero-free functions satisfying the above interpolation conditions can in general be empty, so we will assume in what follows that this set is non void. Concerning existence of extremals, we have:

Theorem 2.2. *The infimum in Problem* 2.1 *is attained.*

Proof. (The following argument is well known and is included for completeness.) Pick a sequence f_k of zero-free functions in A^p such that $l_i(f_k) = c_i$ for every $1 \leq i \leq n$ and every $k = 1, 2, \ldots$, and such that $\|f_k\|_{A^p} \to \lambda$ as $k \to \infty$. Since these norms are bounded, there exists a subsequence $\{f_{k_j}\}$ and an analytic function f such that $f_{k_j} \to f$ as $j \to \infty$. By Hurwitz' theorem, f is zero-free. Moreover, $l_i(f) = c_i$ for every $1 \leq i \leq n$. By Fatou's lemma,

$$(\int_{\mathbb{D}} |f|^p dA)^{\frac{1}{p}} \leq \lambda,$$

but by minimality of λ, we must actually have equality. Therefore f is extremal for Problem 2.1. $\square$

Let us now consider the special case of point evaluation. More specifically, let $\beta_1, \ldots, \beta_n \in \mathbb{D}$ be distinct points and let $l_i(f) = f(\beta_i)$, for $1 \leq i \leq n$. We will assume that none of the c_i is zero.

The following result shows that we need only solve the extremal problem in A^2 in order to get a solution in every A^p space ($p > 0$.)

Theorem 2.3. *If g is minimal for the problem*

$$\inf\{\|g\|_{A^2} : g \text{ is zero-free, } l_i(g) = b_i, 1 \leq i \leq n\},$$

where the b_i are elements of D, then $g^{\frac{2}{p}}$ is minimal for the problem

$$(*) \inf\{\|f\|_{A^p} : f \text{ is zero-free }, l_i(f) = c_i, 1 \leq i \leq n\},$$

where $c_i = l_i(g^{\frac{2}{p}})$.

Proof. The function $g^{\frac{2}{p}}$ is zero-free and

$$\int_{\mathbf{D}} (|g(z)|^{\frac{2}{p}})^p dA(z) = \int_{\mathbf{D}} |g(z)|^2 dA(z) < \infty,$$

so $g^{\frac{2}{p}}$ is in A^p. Moreover by definition, $g^{\frac{2}{p}}$ satisfies the interpolation conditions $c_i = l_i(g^{\frac{2}{p}})$.

Now suppose that $g^{\frac{2}{p}}$ is not minimal for the problem **(*)**. Then there exists $h \in A^p$ zero-free such that $c_i = l_i(h)$ and

$$\int_{\mathbf{D}} |h(z)|^p dA(z) < \int_{\mathbf{D}} |g(z)|^2 dA(z).$$

The function $h^{\frac{p}{2}}$ is a zero-free A^2 function such that

$$\|h^{\frac{p}{2}}\|_2 < \|g\|_2.$$

Moreover

$$l_i(h^{\frac{p}{2}}) = h^{\frac{p}{2}}(\beta_i) = c_i^{\frac{p}{2}} = (g^{\frac{2}{p}}(\beta_i))^{\frac{p}{2}} = g(\beta_i) = b_i.$$

This contradicts the minimality of g for the A^2 problem. $\square$

Notice that by the same argument, the converse also holds; in other words, if we can solve the extremal problem in A^p for some $p > 0$, then we can also solve the extremal problem in A^2. Therefore for the remainder of the paper, we will consider only the case $p = 2$. Notice that if we consider Problem (1.2) without the restriction that f must be zero-free, the solution is very simple and well known. Considering for simplicity the case of distinct β_j, the unique solution is the unique linear combination of the reproducing kernels $k(., \beta_j)$ satisfying the interpolating conditions, where

$$k(z, w) := 1/(1 - \bar{w}z)^2.$$

Since our functions are zero-free, we will rewrite a function f as $f(z) = \exp(\varphi(z))$, and solve the problem (relabelling the c_i)

$$\lambda = \inf\{\| \exp(\varphi(z))\|_{A^2} : \varphi(\beta_i) = c_i, 1 \le i \le n\}. \tag{2.1}$$

Theorem 2.4. *The extremal solution to Problem* (2.1) *is unique.*

Proof. Suppose φ_1 and φ_2 are two extremal solutions to (2.1), that is

$$\lambda = \|e^{\varphi_1}\|_{A^2} = \|e^{\varphi_2}\|_{A^2}$$

and

$$\varphi_1(\beta_i) = \varphi_2(\beta_i) = c_i$$

for every $1 \le i \le n$. Consider

$$\varphi(z) = \frac{\varphi_1(z) + \varphi_2(z)}{2}.$$

This new function satisfies $\varphi(\beta_i) = c_i$ for every $1 \le i \le n$, and therefore

$$\begin{aligned} \lambda^2 &\le \int_{\mathbb{D}} |e^{\varphi(z)}|^2 dA(z) \\ &= \int_{\mathbb{D}} |e^{\varphi_1(z)}||e^{\varphi_2(z)}| dA(z) \\ &\le \|e^{\varphi_1}\|_{A^2}\|e^{\varphi_2}\|_{A^2} \text{ (by the Cauchy-Schwarz inequality)} \\ &= \lambda^2. \end{aligned}$$

This implies that

$$|e^{\varphi_1(z)}| = C|e^{\varphi_2(z)}|$$

for some constant C. Since the function $e^{\varphi_1}/e^{\varphi_2}$ has constant modulus, it is a constant, which must equal 1 because of the normalization. The extremal solution to (2.1) is therefore unique. □

Remark. We can generalize this theorem to some other linear functionals l_i. For instance, one may wish to consider linear functionals l_{ij}, $i = 1, \ldots, n$, $j = 0, \ldots, k_i$, that give the jth Taylor coefficients of f at β_i.

The next three lemmas are the technical tools needed to address the issue of the regularity of the extremal function: we want to show that the extremal function is actually a Hardy space function.

For integers $m \geq n$, consider the class P_m of polynomials p of degree at most m such that $p(\beta_i) = c_i$ for every $1 \leq i \leq n$. Let

$$\lambda_m = \inf\{\|e^{p(z)}\|_{A^2} : p \in P_m\}. \tag{2.2}$$

Lemma 2.5. $$\lim_{m\to\infty} \lambda_m = \lambda.$$

Proof. Notice that λ_m is a decreasing sequence of positive numbers bounded below by λ, so

$$\lim_{m\to\infty} \lambda_m \geq \lambda.$$

On the other hand, let φ^* be the extremal function for (2.1). Write

$$\varphi^*(z) = L(z) + h(z)g(z),$$

where L is the Lagrange polynomial taking value c_i at β_i, namely

$$L(z) = \sum_{i=1}^{n} c_i \frac{\prod_{k=1,k\neq i}^{n}(z-\beta_k)}{\prod_{k=1,k\neq i}^{n}(\beta_i-\beta_k)},$$

$h(z) = \prod_{i=1}^{n}(z-\beta_i)$, and g is analytic in $\mathbb{D}$. For each $0 < r < 1$, define

$$\varphi_r(z) := \varphi^*(rz).$$

Let $\varepsilon > 0$. Notice that there exists $\delta > 0$ such that if $\tilde{c}_i$ are complex numbers satisfying $|c_i - \tilde{c}_i| < \delta$ for $i = 1, \ldots, n$, then $|L(z) - \tilde{L}(z)| < \varepsilon$ (for every $z \in \mathbb{D}$), where $\tilde{L}$ is the Lagrange polynomial with values $\tilde{c}_i$ at β_i. We now pick r close enough to 1 so that

$$\|e^{\varphi^*} - e^{\varphi_r}\|_{A^2} < \varepsilon$$

and

$$|\varphi_r(\beta_i) - \varphi^*(\beta_i)| < \frac{\delta}{2} \quad \text{for } i = 1, \ldots, n.$$

Define $p_{m,r}$ to be the mth partial sum of the Taylor series of φ_r. Given any integer $N \geq n$, pick $m \geq N$ such that

$$\|e^{p_{m,r}(z)} - e^{\varphi_r(z)}\|_{A^2} < \varepsilon$$

and

$$|p_{m,r}(\beta_i) - \varphi_r(\beta_i)| < \frac{\delta}{2} \quad \text{for } i = 1, \ldots, n.$$

Let $\tilde{c}_i = p_{m,r}(\beta_i)$ for $i = 1, \ldots, n$ and let $\tilde{L}$ be the Lagrange polynomial taking values $\tilde{c}_i$ at β_i. Then we can write

$$p_{m,r}(z) = \tilde{L}(z) + h(z)q_{m-n,r}(z),$$

where $q_{m-n,r}$ is a polynomial of degree at most $m-n$. Notice that since $|p_{m,r}(\beta_i) - \varphi(\beta_i)| < \delta$ (for every $i = 1, \ldots, n$),

$$|L(z) - \tilde{L}(z)| < \varepsilon \quad \text{for every } z \in \mathbb{D}.$$

Define
$$p_m(z) = L(z) + h(z)q_{m-n,r}(z).$$
Then $p_m \in P_m$, and
$$\begin{aligned} |e^{p_m(z)} - e^{p_{m,r}(z)}|^2 &\leq |e^{p_{m,r}(z)}|^2(e^{|p_{m,r}(z)-p_m(z)|} - 1)^2 \\ &= |e^{p_{m,r}(z)}|^2(e^{|\tilde{L}(z)-L(z)|} - 1)^2 \\ &\leq |e^{p_{m,r}(z)}|^2(e^{\varepsilon} - 1)^2 \end{aligned}$$
Therefore
$$\begin{aligned} \|e^{p_m} - e^{p_{m,r}}\|_{A^2} &\leq \|e^{p_{m,r}}\|_{A^2}(e^{\varepsilon} - 1) \\ &\leq C(e^{\varepsilon} - 1), \end{aligned}$$
where C is a constant depending only on $\|e^{\varphi^*}\|_{A^2}$. Therefore
$$\|e^{p_m(z)} - e^{\varphi^*(z)}\|_{A^2} \leq 2\varepsilon + C(e^{\varepsilon} - 1) = C_{\varepsilon},$$
which implies
$$\lambda_m \leq \|e^{p_m(z)}\|_{A^2} \leq C_{\varepsilon} + \lambda$$
for arbitrarily large m, where $C_{\varepsilon} \to 0$ as $\varepsilon \to 0$. Therefore
$$\lim_{m\to\infty} \lambda_m \leq \lambda.$$
Since we already have the reverse inequality, we can conclude that
$$\lim_{m\to\infty} \lambda_m = \lambda. \qquad \square$$

Lemma 2.6. *The extremal polynomial p_m^* in (2.2) exists, and for every polynomial ψ_{m-n} of degree at most $m-n$,*
$$\int_{\mathbb{D}} |e^{p_m^*(z)}|^2(z-\beta_1)\ldots(z-\beta_n)\psi_{m-n}(z)dA(z) = 0.$$

Proof. To prove the existence of the extremal polynomial p_m^*, consider the minimizing sequence p_m^k in (2.2). Without loss of generality, we can assume that the functions $e^{p_m^k}$ converge on compact subsets, and hence p_m^k converge pointwise in $\mathbb{D}$ to a polynomial $p_m^* \in P_m$. As above, applying Fatou's lemma, we see that p_m^* is in fact the extremal.

Define
$$F(\varepsilon) = \|\exp(p_m^*(z) + \varepsilon\prod_{i=1}^{n}(z-\beta_i)\psi_{m-n}(z))\|_{A^2}^2$$
where ψ_{m-n} is any polynomial of degree at most $m-n$. Then since p_m^* is extremal, $F'(0) = 0$.
$$\begin{aligned} F(\varepsilon) &= \int_{\mathbb{D}} |\exp(p_m^*(z) + \varepsilon\prod_{i=1}^{n}(z-\beta_i)\psi_{m-n}(z))|^2 dA(z) \\ &= \int_{\mathbb{D}} |\exp(p_m^*(z))|^2 \exp(2\varepsilon Re(\prod_{i=1}^{n}(z-\beta_i)\psi_{m-n}(z))dA(z) \end{aligned}$$

Therefore

$$F'(0) = \int_{\mathbb{D}} |\exp(p_m^*(z))|^2 2Re(\prod_{i=1}^{n}(z-\beta_i)\psi_{m-n}(z))dA(z) = 0.$$

Replacing ψ_{m-n} by $i\psi_{m-n}$ gives

$$\int_{\mathbb{D}} |\exp(p_m^*(z))|^2 2Re(\prod_{i=1}^{n}(z-\beta_i)i\psi_{m-n}(z))dA(z) = 0,$$

and therefore

$$\int_{\mathbb{D}} |\exp(p_m^*(z))|^2 \prod_{i=1}^{n}(z-\beta_i)\psi_{m-n}(z)dA(z) = 0$$

for every polynomial ψ_{m-n} of degree at most $m-n$. □

Lemma 2.7. *For each $m \geq n$, $e^{p_m^*} \in H^2$, and these H^2 norms are bounded.*

Proof. Write

$$p_m^*(z) = L(z) + h(z)q_{m-n}(z),$$

where $L(z)$ is the Lagrange polynomial taking value c_i at β_i (for $i = 1, \ldots, n$), $h(z) = \prod_{i=1}^{n}(z-\beta_i)$, and q_{m-n} is a polynomial of degree at most $m-n$. We then have

$$\begin{aligned} \int_{\mathbb{T}} |e^{p_m^*(e^{i\theta})}|^2 d\theta &= i\int_{\mathbb{T}} |e^{p_m^*(z)}|^2 z d\bar{z} \\ &= 2\int_{\mathbb{D}} \frac{\partial}{\partial z}(|e^{p_m^*(z)}|^2 z)dA(z) \qquad \text{(by Green's formula)} \\ &= \int_{\mathbb{D}} |e^{p_m^*(z)}|^2 (p_m^{*'}(z)z+1)dA(z). \end{aligned}$$

We would like to show that this integral is bounded by $C\|e^{p_m^*(z)}\|_{A^2}^2$, where C is a constant independent of m. First notice that

$$zp_m^{*'}(z) = zL'(z) + zh'(z)q_{m-n}(z) + zh(z)q'_{m-n}(z).$$

Since $zq'_{m-n}(z)$ is a polynomial of degree at most $m-n$, Lemma 2.6 allows us to conclude that

$$\int_{\mathbb{D}} |e^{p_m^*(z)}|^2 zh(z)q'_{m-n}(z)dA(z) = 0.$$

On the other hand, $zL'(z)$ is bounded and independent of m, and therefore

$$|\int_{\mathbb{D}} |e^{p_m^*(z)}|^2 zL'(z)dA(z)| \leq C_1\|e^{p_m^*(z)}\|_{A^2}^2,$$

where C_1 is a constant independent of m. Therefore the crucial term is that involving $zh'(z)q_{m-n}(z)$. Write

$$q_{m-n}(z) = q_{m-n}(\beta_k) + (z-\beta_k)q_{m-n-1}(z),$$

where q_{m-n-1} is a polynomial of degree at most $m-n-1$. Then

$$
\begin{aligned}
zh'(z)q_{m-n}(z) &= z\{\sum_{k=1}^{n}[\prod_{i=1,i\neq k}^{n}(z-\beta_i)]\}\{q_{m-n}(\beta_k)+(z-\beta_k)q_{m-n-1}(z)\} \\
&= \sum_{k=1}^{n}\{z\prod_{i=1,i\neq k}^{n}(z-\beta_i)\}q_{m-n}(\beta_k)+\sum_{k=1}^{n}\{\prod_{i=1}^{n}(z-\beta_i)\}zq_{m-n-1}(z).
\end{aligned}
$$

Since $zq_{m-n-1}(z)$ is a polynomial of degree at most $m-n$, by Lemma 2.6, the contribution of the second big sum above, when integrated against $|e^{p_m^*(z)}|^2$, is zero. On the other hand, it is not hard to see that the polynomials q_{m-n} are (uniformly) bounded on the set $\{\beta_k : k = 1, \ldots, n\}$, and therefore their contribution is a bounded one, that is, there exists a constant C_2 such that

$$\int_{\mathbb{D}} |e^{p_m^*(z)}|^2 zh'(z)q_{m-n}(z)dA(z) \le C_2\|e^{p_m^*(z)}\|_{A^2}^2.$$

We have therefore shown that there exist constants C and M, independent of m, such that

$$\int_{\mathbb{T}} |e^{p_m^*(e^{i\theta})}|^2 d\theta \le C\|e^{p_m^*(z)}\|_A^2 = C\lambda_m \le CM.$$

Thus the functions $e^{p_m^*}$ have uniformly bounded H^2 norms. □

Theorem 2.8. $e^{\varphi^*} \in H^2$.

Proof. By an argument similar to that of Theorem 2.2 and by uniqueness of the extremal function for (2.1), there exists a subsequence $\{p_{m_k}^*\}$ of $\{p_m\}$ such that

$$e^{p_{m_k}^*} \to e^{\varphi^*}$$

pointwise as $k \to \infty$. For each fixed radius r, $0 < r < 1$, by Fatou's lemma,

$$\int_{\mathbb{T}} |\exp(\varphi^*(re^{i\theta}))|^2 d\theta \le \liminf_{k\to\infty} \int_{\mathbb{T}} |\exp(p_{m_k}^*(re^{i\theta}))|^2 d\theta.$$

By Lemma 2.7, the right-hand side is bounded for all $0 < r < 1$, and therefore $e^{\varphi^*} \in H^2$. □

The following corollary follows from Theorems 2.8 and 2.3.

Corollary 2.9. *Let $0 < p < \infty$, and let e^{φ^*} be the extremal function that minimizes the norm*

$$\lambda = \inf\{\|\exp(\varphi(z))\|_{A^p} : \varphi(\beta_i) = c_i, 1 \le i \le n\}.$$

Then e^{φ^*} is in H^p.

3. Another approach to regularity

In the following, we present a very different approach to showing the a priori regularity of the extremal function. It was developed by D. Aharonov and H.S. Shapiro in 1972 and 1978 in two unpublished preprints ([AhSh1, AhSh2]) in connection with their study of the minimal area problem for univalent and locally univalent functions. See also [ASS1, ASS2].

Given n points $\beta_1, \dots, \beta_n$ of $\mathbb{D}$, and complex numbers $c_1, \dots, c_n$ recall that L denotes the unique (Lagrange interpolating) polynomial of degree at most $n-1$ satisfying

$$L(\beta_j) = c_j, j = 1, 2, \dots, n. \tag{3.1}$$

As above, the polynomial h is defined by

$$h(z) := (z - \beta_1) \dots (z - \beta_n).$$

We are considering, as before, Problem (1.2) when the functionals l_i are point evaluations at β_i, in A^2.

Recall that in order to get a nonvacuous problem, we assume that none of the c_j is zero. For a holomorphic function f in $\mathbb{D}$, let $L(f)$ denote the unique polynomial of degree at most $n-1$ satisfying (3.1), with $c_j := f(\beta_j)$. Then, there is a unique function g analytic in $\mathbb{D}$ such that

$$f = hg + L(f).$$

Of course, $L(f)$ is bounded on $\mathbb{D}$ by $C \max |f(\beta_j)|$, where C is a constant depending on the $\{\beta_j\}$ and the $\{c_j\}$, but not on f.

Suppose now for each s in the interval $(0, s_0)$, a_s denotes a univalent function in $\mathbb{D}$ satisfying

$$a_s(0) = 0 \quad \text{and} \tag{3.2}$$

$$|a_s(z)| < 1 \quad \text{for } z \in \mathbb{D}. \tag{3.3}$$

(Thus, by Schwarz' lemma,

$$|a_s(z)| \leq |z| \quad \text{for } z \in \mathbb{D}.)$$

Let G_s denote the image of $\mathbb{D}$ under the map $z \to a_s(z)$.

Let now f be an extremal function for Problem (1.2), that is, it is a zero-free function in A^2 satisfying the interpolating conditions

$$f(\beta_j) = c_j, \quad j = 1, \dots, n, \tag{3.4}$$

and having the least norm among such functions. Then, denoting

$$g_s(z) := f(a_s(z))a_s'(z), \tag{3.5}$$

we observe that the function f_s defined by

$$f_s(z) := g_s(z)L(f/g_s)(z) \tag{3.6}$$

is in A^2 and satisfies the interpolating conditions, since

$$f_s(\beta_j) = g_s(\beta_j)[f(\beta_j)/g_s(\beta_j)] = f(\beta_j).$$

Moreover, g_s is certainly zero-free, and hence so is f_s if we can verify that the polynomial $L(f/g_s)$ has no zeros in $\mathbb{D}$.

Now, we shall impose some further restrictions on the maps a_s. We assume that

$$|a_s(z) - z| \leq B(z)c(s) \text{ and} \tag{3.7}$$

$$|a'_s(z) - 1| \leq B(z)c(s) \tag{3.8}$$

where B is some positive continuous function on $\mathbb{D}$, and c is a continuous function on $(0, s_0]$ such that

$$c(s) \to 0 \text{ as } s \to 0. \tag{3.9}$$

With these assumptions, $a_s(z) \to z$ and $a'_s(z) \to 1$ for each z in $\mathbb{D}$, as $s \to 0$. Thus, $f(\beta_j)/g_s(\beta_j) \to 1$ as $s \to 0$, for each j. Thus, the polynomials

$$L_s := L(f/g_s)$$

of degree at most $n-1$ tend to 1 on the set $\{\beta_1, \ldots, \beta_n\}$ as $s \to 0$, and hence they tend uniformly to 1 on $\mathbb{D}$. It follows that for s sufficiently near 0, L_s has no zeros in $\mathbb{D}$, and consequently f_s is zero-free.

Hence, for sufficiently small s, say $s < s_1$, f_s is a "competing function" in the extremal problem, and we have:

$$\|f\|_{A^2} \leq \|f_s\|_{A^2}. \tag{3.10}$$

Note that $L(f/g_s)$ differs from 1, uniformly for all z in $\mathbb{D}$, by a constant times the maximum of the numbers

$$\{|(f(\beta_j)/g_s(\beta_j)) - 1|,\ j = 1, 2, \ldots, n\}. \tag{3.11}$$

Now,

$$f(z)/g_s(z) - 1 = (f(z) - g_s(z))/g_s(z)$$

and since

$$|g_s(z)| = |f(a_s(z)||a'_s(z)| \to |f(z)| \text{ as } s \to 0,$$

by virtue of (3.7), (3.8), and (3.9) the numbers $|g_s(\beta_j)|$ remain greater than some positive constant as $s \to 0$. Consequently, the numbers (3.11) are, for small s, bounded by a constant times the maximum of the numbers

$$\{|f(\beta_j) - g_s(\beta_j)|,\ j = 1, 2, \ldots, n\}. \tag{3.12}$$

But,

$$\begin{aligned} |f(z) - g_s(z)| &= |f(z) - f(a_s(z))a'_s(z)| \\ &\leq |f(z) - f(a_s(z))| + |f(a_s(z))||1 - a'_s(z)| \end{aligned}$$

Using the estimates (3.7), (3.8) we find that the numbers (3.12) are bounded by a constant times $c(s)$, and therefore

$$L(f/g_s) = 1 + O(c(s)),$$

uniformly for z in $\mathbb{D}$, as $s \to 0$. Hence, from (3.10) and (3.6),

$$\|f\|_{A^2} \leq \|g_s\|_{A^2}(1 + Mc(s))$$

for some constant M, thus

$$\int_{\mathbb{D}} |f(z)|^2 dA \leq \int_{\mathbb{D}} |f_s(z)|^2 dA + Nc(s)$$

for some new constant N. Since

$$\begin{aligned}\int_{\mathbb{D}} |f_s(z)|^2 dA &= \int_{\mathbb{D}} |f(a_s(z)|^2 |a_s'(z)|^2 dA \\ &= \int_{G_s} |f(z')|^2 dA(z') \text{ (changing variables by } z' = a_s(z))\end{aligned}$$

and combining the two integrals yields:

(∗) *Under the assumptions made thus far, the area integral of* $|f|^2$ *over the domain* D_s *complementary to* $G_s = a_s(\mathbb{D})$ *in* $\mathbb{D}$ *does not exceed* $Nc(s)$*, where* N *is a constant and* $c(s)$ *is as in* (3.7) *and* (3.8).

To see the usefulness of **(∗)**, let us first consider an almost trivial choice of a_s, namely

$$a_s(z) = (1-s)z \text{ and } a_s'(z) = 1-s.$$

Then, (3.7) and (3.8) hold with $c(s) = s$. Here G_s is the disk $\{|z| < 1-s\}$, so **(∗)** asserts (denoting $t := 1-s$): the integral of $|f|^2$ over the annulus $\{t < |z| < 1\}$, for all t sufficiently close to 1, is bounded by a constant times $1-t$. Consequently, the mean value of $|f|^2$ over these annuli remains bounded. This, however, easily implies that f is in the Hardy class H^2 of the disk! So, we have given another proof of Theorem 2.8: extremals for the zero-free A^2 problem (1.2) always belong to H^2.

We can extract a bit more, namely that extremals are bounded in $\mathbb{D}$, with a more recondite choice of $a(s)$.

Let w denote a point of the unit circle $\mathbb{T}$, and s a small positive number. Let $G_{s,w}$ denote the crescent bounded by $\mathbb{T}$ and a circle of radius s internally tangent to $\mathbb{T}$ at w. (This circle is thus centered at $(1-s)w$.) Let $a_{s,w}$ be the unique conformal map of $\mathbb{D}$ onto $G_{s,w}$ mapping 0 to 0 and the boundary point w to $(1-2s)w$, and $b_{s,w}$ the z-derivative of $a_{s,w}$. We are going to show

Lemma 3.1. *With* $a_{s,w}$ *and* $b_{s,w}$ *in place of* a_s, a_s' *respectively,* (3.7) *and* (3.8) *hold, with* $c(s) = s^2$*, uniformly with respect to* w.

Assuming this for the moment, let us show how the boundedness of extremals follows. Applying **(*)**, we see that if f is extremal, the area integral of $|f|^2$ over the disk centered at $(1-s)w$ of radius s does not exceed a constant (independent of w and s) times the area of this disk. Since $|f((1-s)w)|^2$ does not exceed the areal mean value of $|f|^2$ over this disk, we conclude $|f((1-s)w)|$ is bounded uniformly for all w in $\mathbb{T}$ and sufficiently small s, i.e., $|f|$ is bounded in some annulus $\{1-s_0 < |z| < 1\}$, and hence in $\mathbb{D}$. We therefore have the following:

Theorem 3.2. *The extremal function* f^* *for Problem* 2.1 *is in* H^∞.

It only remains to prove Lemma 3.1.

Proof. First note that the arguments based on (3.7 and 3.8) leading to **(*)** only rely on the boundedness of the function B on compact subsets of $\mathbb{D}$, and more precisely on a compact subset containing all interpolation points β_j, $j = 1, \ldots, n$. Since clearly (3.8) follows (with a different choice of $B(.)$) from (3.7), we have only to verify (3.7). Also, by symmetry, it is enough to treat the case $w = 1$. We do so, and for simplicity denote $a_{s,1}$, $G_{s,1}$ by a_s, G_s respectively. Thus, a_s maps $\mathbb{D}$ onto the domain bounded by $\mathbb{T}$ and the circle of radius s centered at $1 - s$. Moreover $a_s(0) = 0$, and $a_s(1) = 1 - 2s$. Thus, we have a Taylor expansion

$$a_s(z) = c_{1,s}z + c_{2,s}z^2 + \cdots$$

convergent for $|z| < 1$. Moreover, it is easy to see from the symmetry of G_s that all the coefficients $c_{j,s}$ are real.

Under the map $Z = 1/(1-z)$, G_s is transformed to a vertical strip S in the Z plane bounded by the lines $\{Re\, Z = 1/2\}$ and $\{Re\, Z = 1/2s\}$. Thus, the function

$$h_s := 1/(1 - a_s)$$

maps $\mathbb{D}$ onto S and carries 0 into 1, and the boundary point 1 to ∞. Hence $u_s(e^{it}) := Re(h_s(e^{it}))$ satisfies

$$\begin{aligned} u_s(e^{it}) &= 1/2 \text{ for } |t| > t_0, \text{ and} \\ &= 1/2s \text{ for } |t| < t_0, \end{aligned}$$

where $t_0, 0 < t_0 < \pi$ is determined from

$$1 = \frac{1}{2\pi}\int_{\mathbb{T}} u_s(e^{it}dt = \frac{1}{2} + \frac{1-s}{2\pi s}t_0$$

hence

$$t_0 = (s/(1-s))\pi. \tag{3.13}$$

Now, we have a Taylor expansion

$$h_s(z) = 1 + b_{1,s}z + b_{2,s}z^2 + \cdots \tag{3.14}$$

where the $b_{j,s}$ are real, and so determined from

$$u_s(e^{it} = 1 + b_{1,s}\cos t + b_{2,s}\cos 2t + \cdots,$$

i.e.,

$$b_{n,s} = \frac{2}{\pi}\int_0^{\pi} u_s(e^{it})\cos(nt)dt$$

hence

$$b_{n,s} = \sin nt_0/nt_0 \qquad n = 1, 2, \ldots \tag{3.15}$$

where t_0 is given by (3.13).

We are now prepared to prove Lemma 3.1, i.e.,

$$|a_s(z) - z| \le B(z)s^2. \tag{3.16}$$

We have

$$h_s(z) - \frac{1}{1-z} = \frac{1}{1-a_s(z)} - \frac{1}{1-z} = \frac{a_s(z) - z}{(1-z)(1-a_s(z))},$$

so

$$|a_s(z) - z| \leq 4|h_s(z) - \frac{1}{1-z}| \leq 4\sum_{n=1}^{\infty} |b_{n,s} - 1||z|^n. \tag{3.17}$$

But, from (3.15)

$$|b_{n,s} - 1| = |\frac{\sin nt_0}{nt_0} - 1|.$$

Since the function

$$\frac{(\sin x)/x - 1}{x^2}$$

is bounded for x real, we have for some constant N :

$$|\frac{\sin nt_0}{nt_0} - 1| \leq N(nt_0)^2 \leq N'n^2s^2$$

for small s, in view of (3.13), where N' is some new constant. Thus, finally, inserting this last estimate into (3.17),

$$|a_s(z) - z| \leq N''s^2B(z),$$

where

$$B(z) := \sum_{n=1}^{\infty} n^2|z|^n,$$

which is certainly bounded on compact subsets of $\mathbb{D}$, and the proof is finished. □

Remark. This type of variation can be used to give another proof of the regularity and form of extremal functions in the non-vanishing H^p case, which were originally established in [Kh1, Kh2]. In what follows, we shall only discuss the case $p = 2$, since the case of other p follows at once via an analogue of Theorem 2.3 in the H^p setting.

For the sake of brevity, we only consider the following problem. Given complex constants $c_0, c_1, \ldots, c_m$ with c_0 not zero (w.l.o.g. we could take $c_0 = 1$), let A be the subset of H^2 consisting of "admissible functions" f, i.e., those functions zero-free in $\mathbb{D}$ whose first $m+1$ Taylor coefficients are the c_j. We consider the extremal problem , to minimize $\|f\|_2 := \|f\|_{H^2}$ in the class A. The following argument is again an adaptation of a variational argument used by Aharonov and Shapiro in ([AhSh1, AhSh2]) for a different problem.

Proposition 3.3. *Every extremal is in the Dirichlet space, that is, satisfies*

$$\int_{\mathbb{D}} |f'(z)|^2 dA < \infty.$$

Proof. Let f be extremal, and $0 < t < 1$. Then,

$$f(z) = tf(tz)[f(z)/tf(tz)] = tf(tz)[S(z;t) + R(z;t)] \tag{3.18}$$

where S denotes the partial sum of order m of the Taylor expansion of $f(z)/tf(tz)$ $=: E(z;t)$ and R denotes the remainder $E - S$. Now,

$$|f(z) - f(tz)| \leq C(1-t),$$

uniformly for $|z| \leq 1/2$, where C is a constant depending on f, and this implies easily

$$|1 - E(z;t)| \leq C(1-t)$$

for those z, and some (different) constant C. From this it follows easily that

$$S(z;t) = 1 + O(1-t), \text{ uniformly for } z \in \mathbb{D}. \tag{3.19}$$

Moreover, from (3.18) we see that $tf(tz)S(z;t)$ has the same Taylor coefficients as f, through terms of order m. Also, (3.19) shows that S does not vanish in $\mathbb{D}$ for t near 1. We conclude that, for t sufficiently close to 1, $tf(tz)S(z;t)$ is admissible, and consequently its norm is greater than or equal to that of f, so we have

$$\int_{\mathbb{T}} |f(e^{is})|^2 ds \leq (\int_{\mathbb{T}} |tf(te^{is})|^2 ds)(1 + O(1-t)),$$

or, in terms of the Taylor coefficients a_n of f,

$$\sum |a_n|^2 \leq [\sum |a_n|^2 t^{2n+2}](1 + O(1-t))$$

so

$$\sum (1 - t^{2n+2})/(1-t)|a_n|^2$$

remains bounded as $t \to 1$, which implies f has finite Dirichlet integral. □

Corollary 3.4. *The extremal must be a polynomial of degree at most m times a singular function whose representing measure can only have atoms located at the zeros on $\mathbb{T}$ of this polynomial.*

Proof. As usual, for every $h \in H^\infty$, $(1 + wz^{m+1}h)f$ (where f is extremal, and w a complex number) is admissible for small $|w|$. Hence, as in the proof of Lemma 2.6, we obtain that f is orthogonal (in H^2!) to $z^{m+1}hf$. If $f = IF$, where I is a singular inner function and F is outer, since $|I| = 1$ a.e. on $\mathbb{T}$, it follows that F is orthogonal to $z^{m+1}FH^\infty$. Now, F is cyclic, so FH^∞ is dense in H^2, i.e., F is orthogonal to $z^{m+1}H^2$. Hence, F is a polynomial of degree at most m. For the product FI to have a finite Dirichlet norm, the singular measure for I must be supported on a subset of the zero set of F on $\mathbb{T}$ as claimed. Indeed, for any singular inner function I and any point $w \in \mathbb{T}$ where the singular measure for I has infinite Radon-Nikodym derivative with respect to Lebesgue measure,

$$\int_{\mathbb{D} \cap \{|z-w|<c\}} |I'|^2 dA = \infty,$$

because the closure of the image under I of any such neighborhood of w is the whole unit disk (cf. [CL, Theorem 5.4]). □

4. A discussion of the conjectured form of extremal functions

In this section we provide certain evidence in support of our overall conjecture and draw out possible lines of attack that would hopefully lead to a rigorous proof in the future. Recall that the extremal function f^* in the problem (2.1):

$$\lambda = \inf\{\|\exp(\varphi(z))\|_{A^2} : \varphi(\beta_i) = c_i, 1 \le i \le n\}$$

is conjectured to have the form (1.3):

$$f^*(z) = C\frac{\prod_{j=1}^{2n-2}(1-\bar{\alpha}_j z)^{\frac{2}{p}}\exp(\sum_{j=1}^{k}\lambda_j\frac{e^{i\theta_j}+z}{e^{i\theta_j}-z})}{\prod_{j=1}^{n}(1-\bar{\beta}_j z)^{\frac{4}{p}}},$$

where C is a constant, $|\alpha_j| \le 1$, $j = 1, \dots, 2n-2$, $|\beta_j| < 1$, $j = 1, \dots, n$, $\lambda_j \le 0$, $j = 1, \dots k$, $k \le 2n-2$. As in the previous sections, we shall focus the discussion on the case $p = 2$, since the A^p extremals are simply the $2/p$th powers of those in A^2.

First, let us observe that if the solution to the problem for $p = 2$ in the whole space A^2, i.e.,

$$\lambda = \inf\{\|f(z)\|_{A^2} : f(\beta_j) = \exp(c_j), 1 \le j \le n\} \tag{4.1}$$

happens to be non-vanishing in $\mathbb{D}$, then it solves Problem (2.1.) The solution to Problem (4.1) is well known and is equal to a linear combination of the reproducing Bergman kernels at the interpolation points. That is,

$$f^*(z) = \sum_{j=1}^{n}\frac{a_j}{(1-\bar{\beta}_j z)^2}, \tag{4.2}$$

where the a_j are constants, which does have the form (1.3) with singular inner factors being trivial.

Recall that a closed subset K of the unit circle $\mathbb{T}$ is called a Carleson set if

$$\int_{\mathbb{T}} \log \rho_K(e^{i\theta})d\theta > -\infty,$$

where $\rho_K(z) = \text{dist}\,(z, K)$ (cf., e.g., [DS, p. 250].)

Now, if we could squeeze additional regularity out of the extremal function f^* in (2.1),the following argument would allow us to establish most of (1.3) right away. Namely

Theorem 4.1. *Assume that the support of the singular measure in the inner factor of the extremal function f^* in* (2.1) *is a Carleson set. Then the outer part of f^* is as claimed in* (1.3).

Remark. The regularity assumption for the singular factor of f^* is not unreasonable. In fact, some a priori regularity of extremals was the starting point in ([KS]) for the investigation of linear extremal problems in A^p, i.e., Problem 2.1 but without the non-vanishing restriction. There, the authors have been able to achieve the a priori regularity by considering a dual variational problem whose solution satisfied a nonlinear degenerate elliptic equation. Then, the a priori regularity results for solutions of such equations (although excruciatingly difficult) yielded the desired Lipschitz regularity of the extremal functions. Surprisingly, as we show at the end of the paper, even in the simplest examples of problems for non-vanishing functions in A^2, if the extremals have the form (1.3), they fail to be even continuous in the closed disk. This may be the first example of how some extremals in A^p and H^p differ qualitatively. Of course, the extremal functions for Problem 2.1 in the H^p context are all Lipschitz continuous (cf. Corollary 3.4). Unfortunately, in the context of highly nonlinear problems for non-vanishing functions (since the latter do not form a convex set) the direct duality approach fails at once. (Below, however, we will indicate another line of reasoning which may allow one to save at least some ideas from the duality approach.)

Proof. From the results of the previous sections, it follows that

$$f^* = FS, \tag{4.3}$$

where F is outer and S is a singular inner function whose associated measure $\mu \leq 0, \mu \perp d\theta$ is concentrated on the Carleson set K. Note that

$$S'(z) = S(z)\frac{1}{2\pi}\int_0^{2\pi} \frac{2d\mu(\theta)}{(e^{i\theta}-z)^2},$$

where

$$S(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta).$$

So

$$|S'(z)| = O(\rho_K^{-2}(z)), \tag{4.4}$$

where ρ_K is the distance from z to the set K. By a theorem of Carleson (see [DS], p. 250), there exists an outer function $H \in C^2(\bar{\mathbb{D}})$ such that $K \subset \{\zeta \in \mathbb{T} : H^{(j)}(\zeta) = 0, j = 0, 1, 2\}$, and hence

$$H^{(j)}(z) = O(\rho_K(z)^{2-j}), j = 0, 1, 2 \tag{4.5}$$

when $z \to K$. (4.4) and (4.5) yield then that

$$(HS)' = H'S + HS' = hS, \tag{4.6}$$

where $h \in H^\infty(\mathbb{D})$. Recall from our discussions in Sections 2 and 3 that the extremal function f^* must satisfy the following orthogonality condition:

$$\int_{\mathbb{D}} |f^*|^2 \prod_{j=1}^{n}(z-\beta_j)g\,dA = 0 \tag{OC}$$

for all, say, bounded analytic functions g. Rewriting (OC) as

$$0 = \int_{\mathbb{D}} \bar{F}\bar{S}FS\prod_{j=1}^{n}(z-\beta_j)g\,dA \tag{4.7}$$

and noting that F is cyclic in A^2, so that we can find a sequence of polynomials p_n such that $Fp_n \to 1$ in A^2 (F is “weakly invertible” in A^2 in an older terminology), we conclude from (4.7) that $FS = f^*$ is orthogonal to all functions in the invariant subspace $[S]$ of A^2 generated by S that vanish at the points $\beta_1, \beta_2, \dots, \beta_n$. In particular, by (4.6), f^* is orthogonal to all functions $\frac{\partial}{\partial z}(H\prod_{j=1}^n (z-\beta_j)^2 Sg)$ for all polynomials g, i.e.,

$$0 = \int_{\mathbb{D}} \bar{f}^* \frac{\partial}{\partial z}(H \prod_{j=1}^n (z-\beta_j)^2 Sg) dA. \tag{4.8}$$

Applying Green's formula to (4.8), we arrive at

$$0 = \int_{\mathbb{T}} \bar{f}^* H \prod_{j=1}^n (z-\beta_j)^2 Sg d\bar{z} = \int_{\mathbb{T}} \bar{F} H \prod_{j=1}^n (z-\beta_j)^2 g \frac{dz}{z^2}, \tag{4.9}$$

since $|S| = 1$ on $\mathbb{T}$. Finally, since H is outer and hence cyclic in H^2, there exists a sequence of polynomials q_n such that $Hq_n \to 1$ in H^2. Also there exists a sequence of polynomials p_n such that $p_n \to F$ in H^2, so replacing g by $p_n q_n g$, we obtain

$$0 = \int_{\mathbb{T}} |F|^2 \prod_{j=1}^n (z-\beta_j)^2 g \frac{dz}{z^2} \tag{4.10}$$

for all polynomials g. F.&M. Riesz' theorem (cf. [Du, Ga, Ho, Ko]) now implies that

$$|F|^2 = \frac{z^2 h}{\prod_{j=1}^n (z-\beta_j)^2} \text{ a.e. on } \mathbb{T} \tag{4.11}$$

for some $h \in H^1(\mathbb{D})$. The rest of the argument is standard (see for example [Du], Chapter 8.) Since

$$r(z) := \frac{z^2 h(z)}{\prod_{j=1}^n (z-\beta_j)^2} \geq 0$$

on $\mathbb{T}$, it extends as a rational function to all of $\hat{\mathbb{C}}$ and has the form

$$r(z) = C \frac{z^2 \prod_{j=1}^{2n-2} (z-\alpha_j)(1-\bar{\alpha_j} z)}{\prod_{j=1}^n (z-\beta_j)^2 (1-\bar{\beta}_j z)^2} \tag{4.12}$$

where $|\alpha_j| \leq 1$, $j = 1, \dots, 2n-2$, are the zeros of r in $\bar{\mathbb{D}}$ (zeros on $\mathbb{T}$ have even multiplicity) and $C > 0$ is a constant. Thus, remembering that F is an outer function and so

$$\log F(z) = \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |F(e^{i\theta})|^2 d\theta,$$

we easily calculate from (4.11) and (4.12) that

$$F(z) = C \frac{\prod_{j=1}^{2n-2} (1-\bar{\alpha_j} z)}{\prod_{j=1}^n (1-\bar{\beta}_j z)^2}, |\alpha_j| \leq 1, \tag{4.13}$$

as claimed. □

Several remarks are in order.

(i) If the inner part S of f^* is a cyclic vector in A^2, or, equivalently, by the Korenblum-Roberts theorem (see [DS], p. 249), its spectral measure puts no mass on any Carleson set $K \subset \mathbb{T}$, then (4.7) implies right away that f^* is orthogonal to all functions in A^2 vanishing at $\beta_1, \beta_2, \ldots, \beta_n$, and hence

$$f^* = \sum_{j=1}^{n} \frac{a_j}{(1-\bar{\beta}_j z)^2}$$

is a linear combination of reproducing kernels. Thus, we have the corollary already observed in ([AhSh1, AhSh2]):

Corollary 4.2. *If f^* is cyclic in A^2, it must be a rational function of the form* (4.2).

(ii) On the other hand, if we could a priori conclude that the singular part S of f^* is atomic (with spectral measure consisting of at most $2n-2$ atoms), then instead of using Carleson's theorem, we could simply take for the outer function H a polynomial $p \neq 0$ in $\mathbb{D}$ vanishing with multiplicity 2 at the atoms of S. Then following the above argument, once again we arrive at the conjectured form (1.3) for the extremalf^*.

Now, following S.Ya. Khavinson's approach to the problem (2.1) in the Hardy space context (see [Kh2, pp. 88 ff]), we will sketch an argument, which perhaps, after some refinement, would allow us to establish the atomic structure of the inner factor S, using only the a priori H^2 regularity.

For that, define subsets B_r of spheres of radius r in A^2 :

$$B_r := \{f = e^{\varphi} : \|f\|_{A^2} \leq r\},$$

where

$$\varphi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\nu(\theta), \tag{4.14}$$

$$d\nu = \log\rho(\theta)d\theta + d\mu, \tag{4.15}$$

and $\rho \geq 0$, $\rho, \log\rho \in L^1(\mathbb{T})$, $d\mu$ is singular and $d\mu \leq 0$. Consider the map Λ that maps the subsets B_r into $\mathbb{C}^n$, defined by

$$\Lambda(f) = (\varphi(\beta_j))_{j=1}^n.$$

More precisely, each φ is uniquely determined by the corresponding measure ν and vice versa. Hence, Λ maps the set of measures

$$\Sigma_r := \{\nu : \nu = s(\theta)d\theta + d\mu\}$$

satisfying the constraints

$$d\mu \leq 0 \text{ and } d\mu \text{ is singular} \tag{4.16}$$

$$\exp(s(\theta)), s(\theta) \in L^1(\mathbb{T}) \tag{4.17}$$

$$\|\exp(P(d\nu))\|_{L^2(\mathbb{D})} \leq r, \tag{4.18}$$

where

$$P(d\nu)(re^{i\alpha}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\alpha)} d\nu(\theta)$$

is the Poisson integral of ν, into $\mathbb{C}^n$ by

$$\Lambda(\nu) = (S(\nu)(\beta_j))_{j=1}^n.$$

Here

$$S(\nu)(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\nu(\theta) \tag{4.19}$$

stands for the Schwarz integral of the measure ν. Let us denote the image $\Lambda(\Sigma_r)$ in $\mathbb{C}^n$ by A_r. Repeating the argument in [Kh2] essentially word for word, we easily establish that for all $r > 0$, the sets A_r are open, convex, proper subsets of $\mathbb{C}^n$. (Convexity of A_r, for example, follows at once from the Cauchy-Schwarz inequality as in the proof of the uniqueness of f^* in Section 2.) If we denote by $\vec{c} = (c_1, \ldots, c_n)$ the vector of values we are interpolating in (2.1), then the infimum there is easily seen to be equal to

$$r_0 = \inf\{r > 0 : \vec{c} \in A_r\}.$$

Hence, our extremal function f^* (or equivalently $\varphi^* = \log f^*$) corresponds to a measure $d\nu^* \in \Sigma_{r_0}$ for which $\Lambda(\nu^*) \in \partial A_{r_0}$. So, to study the structure of extremal measures ν^* defining the extremals φ^* or $f^* = e^{\varphi^*}$, we need to characterize those $\nu^* \in \Sigma_r : \Lambda(\nu) \in \partial A_r$. From now on, without loss of generality, we assume that $r = 1$ and omit the index r altogether. Let $\vec{w} = (w_1, \ldots, w_n)$ be a finite boundary point of A. Then there exists a hyperplane H defined by $Re \sum_{j=1}^n a_j z_j = d$ such that for all $\vec{z} \in A$,

$$Re \sum_{j=1}^n a_j z_j \le d \quad \text{while} \quad Re \sum_{j=1}^n a_j w_j = d. \tag{4.20}$$

Let ν^* denote a preimage $\Lambda^{-1}(\vec{w})$ in Σ. Using (4.14 and 4.15) we easily rephrase (4.20) in the following equivalent form:

$$\int_{\mathbb{T}} R(e^{i\theta})d\nu(\theta) \le d \tag{4.21}$$

for all ν satisfying (4.16), (4.17) and (4.18), $(r = 1)$ with equality holding for

$$\nu^* = s^* d\theta + d\mu^* \in \Lambda^{-1}(\vec{w}).$$

Then (4.14, 4.15 and 4.19) yield

$$R(e^{i\theta}) = \frac{1}{2\pi} Re(\sum_{j=1}^n a_j \frac{e^{i\theta}+\beta_j}{e^{i\theta}-\beta_j}), \tag{4.22}$$

a rational function with $2n$ poles at $\beta_1, \ldots, \beta_n$ and $1/\bar{\beta}_1, \ldots, 1/\bar{\beta}_n$ that is real-valued on $\mathbb{T}$. Note the following (see [Kh2]):

Claim. *For d in* (4.21) *to be finite for all measures ν satisfying* (4.16), (4.17) *and* (4.18), *it is necessary that $R \ge 0$ on $\mathbb{T}$.*

Indeed, if $R(e^{i\theta})$ (which is continuous on $\mathbb{T}$) were strictly negative on a subarc $E \subset \mathbb{T}$, by choosing $d\nu = sd\theta$ with s negative and arbitrarily large in absolute value on E and fixed on $\mathbb{T} - E$, we would make the left-hand side of (4.21) go to $+\infty$ while still keeping the constraints (4.16), (4.17) and (4.18) intact, thus violating (4.21).

Now, if we knew that $R(e^{i\theta})$ had at least one zero at $e^{i\theta_0}$, we could easily conclude that the extremal measure ν^* in (4.21) can only have an atomic singular part with atoms located at the zeros of $R(e^{i\theta})$ on $\mathbb{T}$. Then, by the argument principle, since $R(e^{i\theta})$ cannot have more than n double zeros on $\mathbb{T}$, the argument sketched in Remark (ii) following Theorem 4.1 establishes the desired form of the extremal function f^*.

To see why a zero of R at $e^{i\theta_0}$ would yield the atomic structure of the singular part $d\mu^*$ of the extremal measure ν^* in (4.16), (4.17) and (4.18), simply note that if μ^* puts any mass on a closed set $E \subset \mathbb{T}$ where $R > 0$, we could replace μ^* by $\mu_1 = \mu^* - \mu^*|_E$ while compensating with a large negative weight at $e^{i\theta_0}$ not to violate (4.18). This will certainly make the integral in (4.21) larger, thus contradicting the extremality of ν^*. Unfortunately, however, we have no control over whether $R(e^{i\theta})$ vanishes on the circle or not, so this reasoning runs aground if we are dealing with (4.21) for $R > 0$ on $\mathbb{T}$. In order to establish the atomic structure of the singular part of the extremal measure ν^* in (4.21) for $R > 0$ on $\mathbb{T}$, we must come up with a variation of ν^* which would increase $\int_{\mathbb{T}} R(e^{i\theta})d\nu(\theta)$ without violating (4.18). This is precisely the turning point that makes problems in the Bergman space so much more difficult than in their Hardy space counterparts. For the latter, if we had simply gotten rid of the singular part μ^* in ν, i.e., divided our corresponding extremal function f^* by a singular inner function defined by μ^*, then we would not have changed the Hardy norm of f^* at all (while we would have dramatically increased the Bergman norm of f^*). This observation in addition to the elementary inequality $u \ln u - u > u \ln v - v$ for any $u, v > 0$, allowed S.Ya. Khavinson (see [Kh2]) to show that in the context of Hardy spaces, when (4.18) is replaced by a similar restriction on the Hardy norm of $\exp(S(\nu))$, if $R > 0$, the extremal measure ν^* is simply a constant times $\log R(e^{i\theta})d\theta$, and an easy qualitative description of extremals follows right away.

Now, in view of the above discussion, we cannot expect that for our problem, when $R > 0$ on $\mathbb{T}$, the extremal measure ν^* in (4.21) satisfying (4.16), (4.17) and (4.18) is absolutely continuous. But where should we expect the atoms of the singular part μ^* of the extremal ν^* to be located? We offer here the following conjecture.

Conjecture. *If $R > 0$ on $\mathbb{T}$, then the singular part μ^* of the extremal measure ν^* in* (4.21) *is supported on the set of local minimum points of R on $\mathbb{T}$.*

In other words, the singular inner part of the extremal function f^* for Problem 2.1 corresponding to the boundary point of A defined by the hyperplane (4.20) is atomic with atoms located at the local minima of R on $\mathbb{T}$.

The conjecture is intuitive in the sense that in order to maximize the integral in (4.21), we are best off if we concentrate all the negative contributions from the singular part of ν at the points where $R > 0$ is smallest. Note that this conjecture does correspond to the upper estimate of the number of atoms in the singular inner part of the extremal function f^* in (1.3). Indeed, R is a rational function of degree $2n$ and hence has $4n - 2$ critical points (i.e., where $R'(z) = 0$) in $\hat{\mathbb{C}}$. Since the number of local maxima and minima of R on $\mathbb{T}$ must be the same (consider $1/R$ instead), we easily deduce that R cannot have more than $2n-2$ local minima (or maxima) on $\mathbb{T}$. (At least two critical points symmetric with respect to $\mathbb{T}$ must lie away from $\mathbb{T}$.)

One possible way to attempt to prove the conjecture using a variation of the extremal measure ν^* in (4.21) might be to divide the function f^* by a function G that would diminish the singular part μ^* of ν^*. Of course, a natural candidate for such a G would be the contractive divisor associated with the invariant subspace $[J]$ in A^2 generated by a singular inner function J built upon a part μ_0 of μ^* such that $\mu_0 \geq \mu^*$ (recall that $\mu^* \leq 0$), such that the support of μ_0 is a subset of the part of the circle that does not contain the local minima of R. Then (cf. [DuKS]) $G = hJ$, where h is a Nevanlinna function and $\|f^*/G\|_{A^2} \leq \|f^*\|$, so (4.18) is preserved. Unfortunately, $|h| > 1$ on $\mathbb{T} - supp(\mu_0)$, so the resulting measure ν defined by $\log(f^*/G) = S(\nu)$ may at least a priori actually diminish the integral in (4.21) instead of increasing it.

Finally, we remark that for the special case when the $\beta_j = 0$ and instead of Problem 2.1 we have the problem of finding

$$\inf\{\|f\|_{A^p} : f \neq 0, f^{(j)}(0) = c_j, j = 0, \dots, n\}, \tag{4.23}$$

the conjectured general form of the extremal function f^* collapses to

$$f^*(z) = C \prod_{j=1}^{n} (1 - \bar{\alpha_j} z)^{\frac{2}{p}} \exp\left(\sum_{j=1}^{k} \lambda_j \frac{e^{i\theta_j} + z}{e^{i\theta_j} - z}\right), \tag{4.24}$$

where $|\alpha_j| \leq 1, j = 1, \dots, n$, $k \leq n$, $\lambda_j \leq 0$. The difference in the degree of the outer part in (4.24) versus the rational function in (1.3) appears if one follows the proof of Theorem 4.1 word for word arriving at

$$|F|^2 = \frac{\prod_{j=1}^{n} (z - \alpha_j)(1 - \bar{\alpha_j} z)}{z^n}$$

instead of (4.12).

We shall discuss Problem 4.23 for $n = 2$ in great detail in the last section.

5. The minimal area problem for locally univalent functions

In this section we shall discuss a particular problem arising in geometric function theory and first studied by Aharonov and Shapiro in [AhSh1, AhSh2]. The problem is initially stated as that of finding

$$\inf\{\int_{\mathbb{D}} |F'(z)|^2 dA : F(0) = 0, F'(0) = 1, F''(0) = b, F'(z) \neq 0 \text{ in } \mathbb{D}\}. \tag{5.1}$$

Problem (5.1) has the obvious geometric meaning of finding, among all locally univalent functions whose first three Taylor coefficients are fixed, the one that maps the unit disk onto a Riemann surface of minimal area. Setting $f = F'$ and $c = 2b$ immediately reduces the problem to a particular example of problems mentioned in (4.23), namely that of finding

$$\inf\{\int_{\mathbb{D}} |f|^2 dA : f \neq 0 \text{ in } \mathbb{D}, f(0) = 1, f'(0) = c\}. \tag{5.2}$$

Assuming without loss of generality that c is real, we find that the conjectured form of the extremal function f in (5.2) is

$$f(z) = C(z - A)e^{\mu_0 \frac{z+1}{z-1}}, \tag{5.3}$$

where $\mu_0 \geq 0$, and C, A, and μ_0 are uniquely determined by the interpolating conditions in (5.2). Of course, if $|c| \leq 1$ in (5.2), the obvious solution is

$$f^* = 1 + cz,$$

and hence, $F^* = z + \frac{c}{2}z^2$ solves (5.1), mapping $\mathbb{D}$ onto a cardioid. The nontrivial case is then when $|c| > 1$. All the results in the previous sections apply, so we know that the extremal for (5.2) has the form

$$f^* = hS,$$

where h is a bounded outer function and S is a singular inner function. As in Section 2, a simple variation gives us the orthogonality conditions (OC) as necessary conditions for extremality:

$$\int_{\mathbb{D}} |f^*|^2 z^{n+2} dA = 0, \; n = 0, 1, 2, \ldots. \tag{5.4}$$

From now on, we will focus on the non trivial case of Problem 5.2 with $c > 1$. Thus, the singular inner factor of f^* is non trivial (cf. Corollary 4.2). In support of the conjectured extremal (5.3), we have the following proposition.

Proposition 5.1. *If the singular factor S of f^* has associated singular measure $d\mu$ that is atomic with a single atom, then*

$$f^*(z) = C(z - 1 - \mu_0)e^{\mu_0 \frac{z+1}{z-1}} \tag{5.5}$$

where C and the weight μ_0 are uniquely determined by the interpolating conditions.

Remark. Although we have been unable to show that the singular inner factor for the extremal f^* is atomic, we offer some remarks after the proof that do support our hypothesis. If this is indeed the case, this would be, to the best of our knowledge, the first example of a "nice" extremal problem whose solution fails to be Lipschitz continuous or even continuous in the closed unit disk. All solutions to similar or even more general problems for non-vanishing H^p functions are Lipschitz continuous in $\bar{D}$ (cf. [Kh2] and the discussion in Section 4). Also, solutions to similar extremal problems in A^p without the non-vanishing restriction are all Lipschitz continuous in $\bar{\mathbb{D}}$ (cf. [KS]).

Proof. Our normalization ($c \in \mathbb{R}^+$) easily implies that the only atom of S is located at 1. So, $f^* = hS$, where S is a one atom singular inner function with mass μ_0 at 1, and h is outer. By Caughran's theorem ([Ca]), the antiderivative F^* of f^* has the same singular inner factor S and no other singular inner factors, i.e.,

$$F^* = HS, \tag{5.6}$$

where H is an outer function times perhaps a Blaschke product. Writing the orthogonality condition (5.4) in the form

$$\int_{\mathbb{D}} \bar{f^*} f^* z^2 p \, dA = 0$$

for any arbitrary polynomial p, and applying Green's formula, we obtain

$$\int_{\mathbb{T}} \bar{F^*} f^* z^2 p \, dz = 0 \tag{5.7}$$

for any arbitrary polynomial p. Using (5.6) and $S\bar{S} = 1$ a.e. on $\mathbb{T}$ yields

$$\int_{\mathbb{T}} \bar{H} h z^3 p \, d\theta = 0. \tag{5.8}$$

Since h is outer, hence cyclic in H^2, we can find a sequence of polynomials q_n such that $hq_n \to 1$ in H^2. Replacing p by $q_n p$ and taking a limit when $n \to \infty$ yields

$$\int_{\mathbb{T}} \bar{H} z^3 p \, d\theta = 0 \tag{5.9}$$

for all polynomials p. This last equation immediately implies that H is a quadratic polynomial. Now, $f^* = hS = (HS)' = H'S + HS'$, and $S' = \frac{2\mu_0}{(z-1)^2} S$. Since $f^* \in H^2$, H must have a double zero at 1 to cancel the pole of S'! Hence $H(z) = C(z-1)^2$, $F(z) = C(z-1)^2 S(z)$, and $f^*(z) = C(z - 1 - \mu_0)\exp(\mu_0 \frac{z-1}{z+1})$ as claimed. □

We want to offer several additional remarks here.

(i) Obviously, the above calculations are reversible, so the function (5.5) does indeed satisfy the orthogonality condition (5.4) for the extremal.

(ii) The proof of Proposition 5.1 can be seen from a slightly different perspective. From Theorem 4.1, it already follows (assuming the hypothesis) that the outer part of f^* is a linear polynomial. Moreover, (5.7) implies that the antiderivative F^* of f^* is a noncyclic vector for the backward shift and hence has a meromorphic pseudocontinuation to $\hat{\mathbb{C}} - \mathbb{D}$ ([DSS]). Accordingly, F^* must be single-valued in a neighborhood of its only singular point $\{1\}$. This implies that $f^* = hS$ must have a zero residue at 1. (Otherwise F would have a logarithmic singularity there.) Calculating the residue of f^* at 1 for a linear polynomial h and an atomic singular factor S yields (5.5).

(iii) The only remaining obstacle in solving the extremal problem (5.2) is showing a priori that the singular inner factor of the extremal function is a one atom singular function. If one follows the outline given in Section 4, we easily find that for the problem (5.2), the function $R(e^{i\theta})$ in (4.22) becomes a rational function of degree 2, and since $R \geq 0$ on $\mathbb{T}$,

$$R(e^{i\theta}) = \text{const}\,\frac{(e^{i\theta} - a)(1 - \bar{a}e^{i\theta})}{e^{i\theta}} = \text{const}\,|e^{i\theta} - a|^2, \tag{5.10}$$

where $|a| \leq 1$. Thus, as we have seen in Section 4, we would be done if we could show that the one atom measure is the solution of the extremal problem

$$\max\{\int_{\mathbb{T}} R(e^{i\theta})d\mu(\theta) : \mu \leq 0, \mu \perp d\theta\} \tag{5.11}$$

where μ satisfies the constraint

$$\int_{\mathbb{D}} |h|^2 |S_\mu|^2 dA \leq 1 \tag{5.12}$$

for a given outer function h and R is given by (5.10). (Recall that S_μ is the singular inner function with associated singular measure μ.) Again, as noted previously, it is almost obvious when $|a| = 1$, since then we simply concentrate as much charge as needed at a to satisfy the constraint without changing the integral (5.11). Yet, in general, we have no control over where in $\mathbb{D}$ a appears.

(iv) Let $k(z)$ denote the orthogonal projection of $|f^*|^2$ onto the space of L^2 integrable harmonic functions in $\mathbb{D}$. The orthogonality condition (5.4) implies that $k(z)$ is a real harmonic polynomial of degree 1. Moreover, due to our normalization of the extremal problem (i.e., $c \in \mathbb{R}$), we can easily show that f^* in fact has real Taylor coefficients. Indeed, $f_1(z) := \overline{f^*(\overline{z})}$ satisfies the same interpolating conditions and has the same L^2-norm over $\mathbb{D}$, thus by the uniqueness of the extremal function, f_1 must be equal to f^*. Since f^* has real Taylor coefficients, the projection of $|f^*|^2$ is an even function of y, and thus

$$k(z) = A + Bx, \tag{5.13}$$

where $A = \int_{\mathbb{D}} |f^*|^2 dA$ and $B = 4\int_{\mathbb{D}} z|f^*|^2 dA$. The orthogonality condition (5.4) now implies that the function $|f^*|^2 - k$ is orthogonal to all real-valued L^2 harmonic functions in $\mathbb{D}$. Using the integral formula in [DKSS2] (or [DS, Chapter 5, Section 5.3]), it follows that

$$\int_{\mathbb{D}} (|f^*(z)|^2 - k(z))s\,dA \geq 0 \tag{5.14}$$

for all functions s that are smooth in $\bar{\mathbb{D}}$ and subharmonic. The following corollary of (5.14) offers an unexpected application of the conjectured form of the extremal f^*.

Corollary 5.2. *Let $w \in \mathbb{T}$, and assume that $|f^*|^2/|z - w|^2 \in L^1(\mathbb{D})$. (Note that the conjectured extremal satisfies this condition at the point $w = 1$.) Then $k(w) \leq 0$. Thus, if f^* has the form* (5.5)*, $B \leq -A$ in* (5.13)*.*

Proof. Choose $s(z) = 1/|rw-z|^2$ for $r > 1$. Applying (5.14) as $r \to 1^+$, we see that if $k(w) > 0$, the integral on the left must tend to $-\infty$, which violates (5.14). □

Calculating the classical balayage $U(e^{i\theta})$ of the density $|f^*|^2 dA$ to $\mathbb{T}$, i.e.,

$$U(e^{i\theta}) = Re\frac{1}{2\pi}\int_{\mathbb{D}} |f^*|^2 \frac{e^{i\theta}+z}{e^{i\theta}-z} dA, \tag{5.15}$$

expanding the Schwarz kernel $\frac{e^{i\theta}+z}{e^{i\theta}-z}$ into the power series with respect to z and using the orthogonality condition (5.4) allows us to cancel all the terms containing powers of z of degree 2 and higher, so we arrive at

$$U(e^{i\theta}) = A + \frac{B}{2}\cos\theta,$$

where A and B are as in (5.13). Since $U > 0$ on $\mathbb{T}$ (it is a "sweep" of a positive measure!), it follows that

Corollary 5.3. $A > \frac{|B|}{2}$, *i.e.,* $\int_{\mathbb{D}} |f^*|^2 dA > 2|\int_{\mathbb{D}} z|f^*|^2 dA|$.

A calculation confirms that for f^* as in (5.5), Corollary 5.3 does hold.

(v) If we denote the value of the minimal area in (5.1) by $A = A(b)$ and by a_3 the coefficient of z^3 in the Taylor expansion of the extremal function F^* (i.e, $F^*(z) = z + bz^2 + a_3 z^3 + \cdots$, where F^* is the anti-derivative of our extremal function f^*) then as was shown in ([AhSh2], Theorem 4, p. 21), the following equality must hold:

$$(3a_3 - 2b^2 - 1)A'(b) + 4bA(b) = 0. \tag{5.16}$$

An involved calculation yields that the conjectured extremal function $F^* = \int f^*$, where f^* is as in (5.5), does indeed satisfy (5.16). This serves as yet one more justification of the conjectured form of the extremal. A number of other necessary properties of the extremal function are discussed in [AhSh1, AhSh2].

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