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Sustainability factors in dynamical systems modeling: Simulating the non-linear aspects of multiple equilibria

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abstract

What is sustainability? Sustainability is a concept that can be defined in many ways depending upon a society's perception of current material needs and the actual material needs of future generations. Much of our ability to achieve sustainability entails developing indicators and measurements that will guide us to this goal. This paper suggests that we can strengthen the prediction of sustainability indicators by adopting a "multiple equilibria" approach for a more effective decision-making process in various sectors of the economy, in ecosystem protection, or in political arenas. There is an emerging need for further development of predictive mathematical models of system sustainability over economic growth models for sustainable resource measurement and management. The objective of this paper is to use computer modeling and differential equations to simulate the "multiple equilibria" of a 3 variable real world system. In our study, we tested the theoretical validity of "multiple equilibria" sustainability modeling through simulated measurements of precipitation and nitrogen runoff into a hypothetical lake. As a quantitative tool to model, the "multiple equilibria" techniques can have tremendous predictive power for business leaders, political decision makers, and environmental scientists, and assist in better management of ecological, economic, and material resources in short-term and long-term end-use scenarios.

1. Introduction: why sustainability is important

Biophysical sustainability is the process of balancing resource stocks and flows within a dynamical system over time. Sustainability is a universal necessity, because, in the natural world, an ecosystem thrives on the symbiotic interaction of numerous individual organisms and communities of organisms that depend on each other biologically and ecologically. Thus, an ecosystem dynamically strives to be in equilibrium but often finds itself far from equilibrium in real world scenarios. But a system's equilibrium can be constrained by its input availability and its output absorbance capabilities. According to Fath (2015), "... meeting Input–Output requirements are necessary but not sufficient conditions for sustainability. For ecosystems, the input constrains are fundamentally energy and matter flows that manifest themselves in terms of solar radiation, global carbon cycle, rate of nitrogen cycling, rate of hydrological cycle, etc. The ability of the environment to accept the system output is constrained by the rate of decomposition, the rate of accumulation of unwanted by-products, and the synergistic couplings that allow material reuse. The adjacent system receiving output must be a lower gradient than the system generating them [making it] necessary for the continual renewal of the configurations that emerge out of these flow gradients" (p. 14). So, internal dynamics in the ecosystem are just as important as external dynamics.

Over time, natural systems either remain sustainable, if they are stable and resilient, or they become unsustainable, if they are fragile or fail to adapt to the dynamics of change. A fragile ecosystem is likely to be an unstable ecosystem due to limited resources or weak symbiotic integrations in the system. If there is an overshoot of population thresholds with persistent nitrogen deficiencies or resource disruptions, fragile system populations will begin to die off and affected species drift toward extinction. When a system is stable and/or resilient it has a capacity to withstand external stress and disturbances, and can quickly recover from systemic shock and return to its original state or an approximate state of functionality.

Lambin (2007) suggests, [an] ecosystem's degree of resilience is often a better indicator of its "health" than its stability. A stable system is often un-resilient because it has rigidly protected itself

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against minor disturbances, rather than develop mechanisms for flexibly coping with major disturbances. In the language of mathematics, the resilience can be described as branching points or bifurcations since when a dynamical system is disturbed it can naturally rebound for better or worse. As an ecosystem evolves, the ecosystem can be acted upon by disturbances knocking it into two different possible states. If both states are stable then the system is robust enough to recover from these external stresses. If not, then the system is said to be unsustainable. To explain the bifurcation process at the macro-scale, Lambin (2007) uses the example of the vast network of dams and sea walls constructed by industrialized countries to protect urban environments from inundation. But, this process has caused natural soil fertilization to be replaced by sizable amounts of chemical fertilizer. Excessive runoff can pollute waterways or lead to eutrophication (or algal blooms) in the regional water systems that possibly feed the red tides and lead to oceanic dead zones. In addition, when flood waters rise to a height where they can overflow the barriers or they can break. Once these events take place, one must address the tremendous economic, ecological and social costs. The aftermath of Hurricanes Rita, Katrina and Sandy are stark examples of how resilience tradeoffs can have devastating impacts. Under normal circumstances strengthened levees and self-sustaining barrier islands, wetlands, and coastal forests would have acted as buffers against the storm surges minimizing environmental damage and human hardship. Therefore, the notion of sustainability is a strategic endeavor and a vast effort to preserve the human condition.

On a societal level, sustainability involves basic life systems, maintenance of diversity, stability in providing goods and services, basic human needs and intangible human needs and support. To reach these objectives, sustainability managers may rely on spatial factors (household, local, regional, national, global), temporal factors (days, months, years, decades), identification of critical sectors (government, industry, community) or resources (natural, synthetic, energy), identification of the characteristics and sensitivities of groups in society (citizens, consumers, cultures), the recognition, creation and maintenance of required organizational and institutional structures, and the degree of risk acceptable in designing sustainable futures (Garner, 2011). In practice, "sustainability" involves these topical considerations, but sustainability indicators and sustainability measurements are also necessary to set goals and determine a relevant course of action. It is the development of sustainability indicators that establishes a baseline for measurement and provides mechanisms for targeted application of sustainable technologies.

2. The Limits to Growth model

Early research on sustainability used the predictive power of computer modeling to simulate how dynamical systems would behave, and eventually brought attention to the stress on natural resources by growing human populations and the limited carrying capacity of the Earth's ecosystems. In the late 1950s, MIT Professor Jay Forrester established the field of "systems dynamics" by using mathematical modeling to analyze the behavior of complex engineering and social systems. Forrester's computer program was designed to simulate a web of complex systems with interactive feedback loops and non-linear equations (Harvey and Hallett, 1977; Jin et al., 1995).

In 1969, Italian business executive Aurelio Peccei published the book, The Chasm Ahead which predicted that civilization will eventually face limitations to population growth, pollution, materials, and energy. Since these problems were global, Peccei believed that these problems should be studied on a global scale. He decided to form an interdisciplinary team of eminent scientists and international consultants into a futurist think tank called the Club of Rome (Humphrey and Buttel, 1982). Professor Forrester's "systems dynamics" modeling methods used extremely complex mathematical equations that seemed an appropriate tool to study the problems envisioned by Peccei. In 1970, Professor Dennis Meadows and a small team of researchers at Massachusetts Institute of Technology (MIT) joined Forrester who was using his modeling methods to support the Club of Rome's Project on the Predicament of Mankind. This collaboration resulted in the report, The Limits to Growth, two years later.

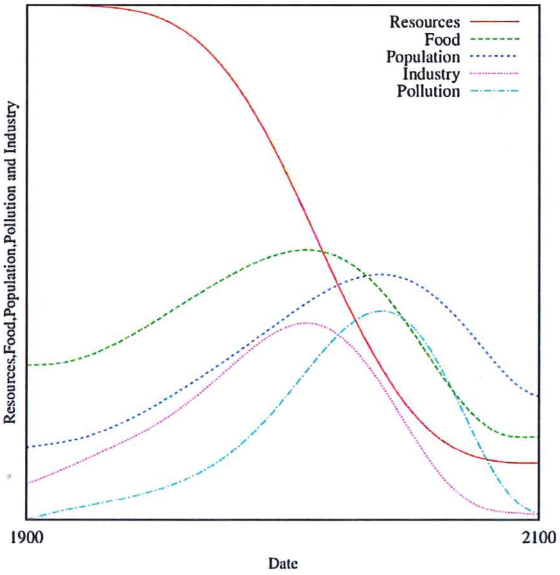

Fig. 1. The standard scenario as reproduced from Meadows, Donella H., Meadows, Dennis L., Randers, Jørgen, and William Behrens III, (1972), Limits to Growth, New York: Universe.

The Limits to Growth report identified the complex web of technical, economic, ecological, social and political problems that all countries face and aggregated them to a global level. The Club of Rome's research team chose five basic quantities whose levels indicated essential components to the state of our world system: population, pollution, natural resources, agricultural capital (or output), and industrial capital (or output). They then established levels and rates of flow along with feedback loops to describe interrelationships among key factors and develop a responsive systems model. Next, the model's mathematical behavior was run through a computer to establish its graphical behavior over the time period 1900–2100 A.D. The computer model produced what was called the World Model Standard Run (Harvey and Hallett, 1977).

According to Meadows et al. (1972), the "standard" world run made no changes in the historical physical, economic and social relationships that governed the development of world systems, so the run plotted the five basic quantities from the years 1900 to 1970. But, the model goes further by using that existing knowledge of current world systems levels to project these operational levels towards the year 2100. The results revealed that if the quantity levels continued to proceed at the current rates, food, industrial output and population would grow exponentially until the rapidly diminishing natural resource base would force a slowdown in industrial growth. While population and pollution will continue to grow for a while after the peak of industrialization, eventually population will start to decline due to increasing death rates once food and medical services decrease as shown in Fig. 1.

Although the "Standard" run was alarming, and perhaps, unrealistic exponential growth, it did take stock of global resource quantities and suggest that there is an opportunity for humans to

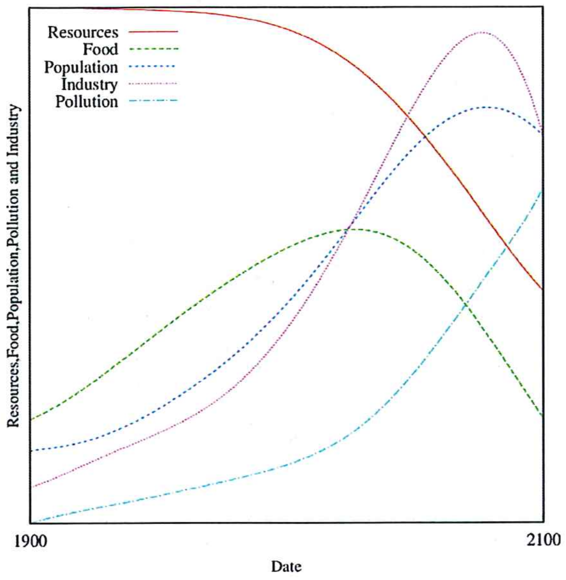

Fig. 2. The optimistic scenario. Reproduced from Meadows, Donella H., Meadows, Dennis L., Randers, Jørgen, and William Behrens III. (1972). Limits to Growth. New York: Universe.

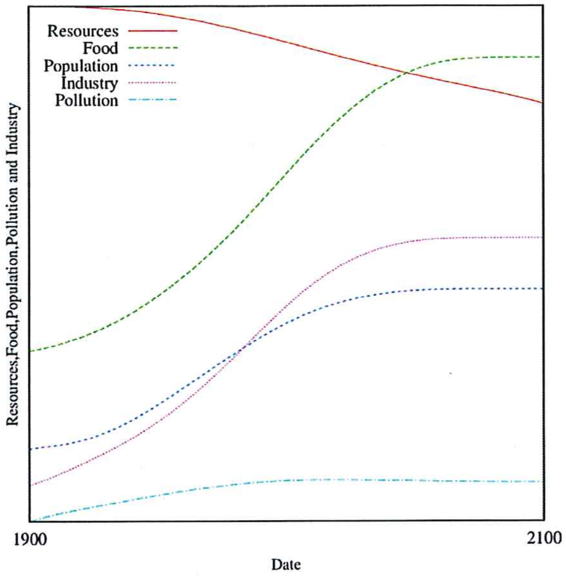

Fig. 3. The steady state scenario. Reproduced from Meadows, Donella H., Meadows, Dennis L., Randers, Jørgen, and William Behrens III. (1972). Limits to Growth. New York: Universe.

adjust our collective behaviors. So, the Club's MIT team ran several alternative simulations of the model. In Fig. 2, these more optimistic runs revealed that if there were unlimited natural resources and a 75%reduction in world pollution, human population would reach a larger size than the "Standard" run, but eventually begin to decrease due to limits to global food supplies as arable land disappears. Yet, even in this optimistic scenario, resources will be depleted in the near future. Ultimately, the MIT team came to the conclusion that future conditions were dire in any scenario the model could produce unless there was an emphasis on achieving balance or equilibrium among the main components of world growth. This condition would be considered a low or no growth economy, and often referred to as the "Steady State" economy where flows of resources into production and pollution of the environment are controlled and stabilized (see Fig. 3).

The Limits to Growth report drew both support for its vision and criticism for its gross generalizations, but in the wake of 1973 and 1979 energy crises initiated by the Organization of Petroleum Exporting Countries (OPEC), the study's warning on global limits to natural resources was made more apparent to the general public. Still, economic growth models and mass consumerism are embedded in the prevailing paradigms of cost-benefit analysis and the discourse of public policy making. Economic growth models such as the Gross Domestic Product (GDP), Consumer Price Index (CPI), and the Stock Exchange are commonly recognized as measurement tools that gauge the health of a "market" economy. But these economic and political infrastructures often fail to recognize the limited access and supply of natural capital and ecological systems as foundations of economic prosperity. Even today it is evident that we live on a planet of finite resources, and even though the rate is slowing, human population growth is still increasing. To have sustainable development into the long-term future there needs to be dramatic and deliberate change in social policy around the world about the ideology of unlimited growth. Ideas about the economic structure of growth and consumption, the technical dynamics of societal change, and the ecological limits of environmental services provide ready-made starting points from which to embark upon a new sustainability agenda.

For many environmentalists, halting economic growth was the key to reversing the suicidal trajectory of the "standard" run toward exponentially increasing resource use. All societies need to have a thriving economic system to remain stable and viable in the long-term. The steady-state economy merely suggest that flows of resources into production and of pollution back into the environmental are kept at a steady level of operation (Costanza et al., 1997). Still, the steady-state economy is more metaphorical than actual. No economy will ever be completely stable and unchanging, but the goal of the steady-state is to create and stay within the parameters of an upper limit and a lower limit of resources use and waste production over a given time period. An inventory of resource stocks and an assessment of periodic and aperiodic use of these resources is needed to design a steady-state economy that will serve current society's immediate needs while planning for the material needs of future generations. Therefore, long-term human sustainability will depend on the global realization that there is an interdependence of all natural resources and that regional sustainability policies will lead to a dynamical stability of the whole Earth system over time, if we seek to establish and sustain a global "steady-state" economy.

The second decade of the 21st century appears to be a social paradigm shifting towards a greener economy and sustainable systems management. Much of this emerging awareness stems from decades of systems modeling that suggest uncontrolled growth in any manner is unsustainable. The Earth is currently a planet with dwindling natural resources, threatened ecosystems, and an exponentially expanding, materials-needy human population. How we collectively address persistent and growing human demand for natural and synthetic resources, economic goods and ecological services, and massive pollution generation will determine the long-term survivability of our species. Much of our ability to achieve sustainability entails developing indicators and measurements that will guide us to this goal. We can strengthen the prediction of sustainability indicators by adopting a "dynamical systems" approach for a more effective decision-making process in various sectors of the economy or in political areas such as environmental policy systems (Saysel et al., 2002). We consider the sustainability dynamic a property of a system which allows for a precise, mathematical definition that draws on the notions of stability and the robustness of "multiple equilibria." Slow or no growth models, as sustainability indicators, will begin to gauge the metabolism of a "green" economy, and offer predictive formats that recognize system efficiencies (or inefficiencies) and recommend necessary adjustments and/or appropriate technologies for economic maintenance.

3. The theory of multiple equilibria

According to Shackley (2000), "[In] a system with many interacting variables and feedbacks, the relative significance of which cannot be assessed a priori, many of which may realistically change on the time and space scales of interest, and which are heterogeneous in the sense that they include a wide range of natural and social processes, computer or simulation modeling sometimes seems the only viable research strategy." This assumption was apparent when we contemplated the approach to be taken in developing sustainability indicators for a troubled and shifting economic system.

There is an emerging need for further development of predictive mathematical models of system sustainability over economic growth models for sustainable resource measurement and management. It is likely that such a stabilizing process would require "multiple equilibria" models of certain sustainability indicators to achieve the proper policy recommendations and technical proficiency. Our goal is to explore the possible existence of multiple steady states through using computer simulations and assuming a dynamical systems approach. This paper confronts the complex interplay of social, economic, and ecological conditions that perpetuate over-accumulation and waste by charting sustainable alternatives through the process of creating "multiple equilibria." Cerny (2010) loosely defines multiple equilibria as "the existence of multiple alternative potential future developmental pathways generated by a [system]." Further, "the effects that generate multiple equilibria create the [mathematical] possibility of new branching points or bifurcations opening the way to potential path modification and reconstruction of the system itself." While the concept of "multiple equilibria" has its roots in macroeconomic theory and political theory (Cerny, 2010), its theoretical basis is grounded on a predictive dynamical systems approach to sustainability (Masson, 1999; Morris and Shin, 2000).

On a practical level, long-term sustainability is a goal that is difficult to achieve without the systemic integration of ecological, economic, and social equity factors since there is a process of cooperative and competing interconnections in dynamical polycentric configurations. The assumption is that a monocentric equilibrium has its drawbacks. The hypothesis is that "monocentricity" may not be a stable system due to uncertain external factors. Consequently, the development of non-monocentric (duocentric, tricentric, polycentric) models of sustainability are needed for theoretical completeness and practical usefulness (Fujita and Ogawa, 1982). Three modeling assumptions simplify this conceptual framework. First, sustainability fundamentals are about establishing and maintaining equilibrium in a system; and second, the larger the system the greater the complexity; and third, complex systems can only be managed with a polycentric multiplicity of equilibria. The challenge for modelers is to identify the essential factors creating equilibrium in all components of a complex system. Further, there may be different types of "equilibria" that can model such complex systems. "Multiple equilibria", much like magnetism, acts like mutual induction to create sustainable systems dynamics (see Fig. 4). This process is similar to "synergism" where positive utility exceeds negative utility because of the preponderance of positive mutualistic relations in the system (Fath, 2004).

4. Modeling multiple equilibria

As a quantitative tool, the goal of modeling the "multiple equilibria" technique is to provide predictive power for business leaders, political decision makers, and environmental scientists, and assist in better management of ecological, economic, and material resources in short-term and long-term sustainability scenarios (Jiang and Shi, 1995). In this section, we use computer modeling to simulate "dynamical systems" of a three (3) variable real world system. We make explicit the simulation of sustainability by providing a hypothetical example of the cyclical rainfall and nitrogen runoff into a typical lake. We used a set of first-order, non-linear differential equations, known collectively as Lotka and Volterras prey–predator models to simulate a dynamical system of relationships among three variables (Maheshwari et al., 2014). These types of equations offer a mathematical description of the cooperation and competition dynamics between species or, in our application, variables in sustainability models (Bennett and Neil, 2013; Bettge, 2009). Our simulation illustrates that the application of nitrogens, the amount of precipitation and the pollution of a lake can be managed sustainability. Each factor can be considered a system component with its own internal balance or equilibrium threshold. Each equilibrium threshold is dependent on its relationship to "equilibrium points" in the systems evolution. Equilibrium points can provide a stability analysis through a fixed point in the system. Any perturbation can lead an equilibrium point from stable to unstable, and vice-versa, depending on the values of the parameters (Maheshwari et al., 2014). Simulating the process of multiple equilibria provides a tricentric predictive tool for modeling sustainability.

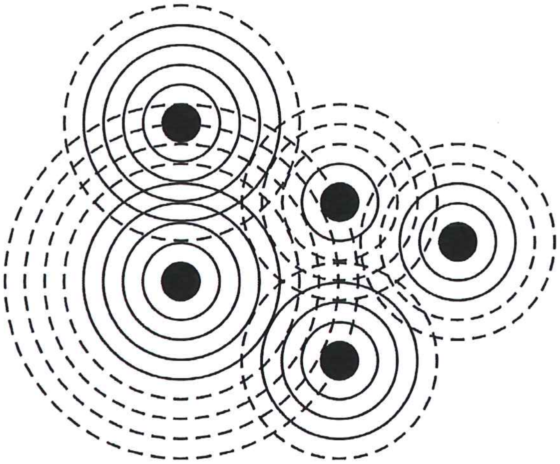

Fig. 4. Polycentric and multiple equilibria. Solid black circles represent multiple equilibria with local effects indicated in solid concentric circles. Dashed concentric circles represent global effect of a multiple equilibria.

All non-linear prey–predator models share an important feature: competition among species whose cooperative behavior can lead to an environment of mutual benefit. These are regions of stability are imposed mathematical conditions on the set of parameters. When these conditions are not satisfied, it can lead to unrealistic or unwanted behavior, such as the extinction of a predator with an exponential growth of prey or predators with an unlimited supply of prey. One type of attractor, the "limit cycle", allows for a manageable dynamical system but we must also include bifurcations if we require robustness or "multiple equilibria" (Nestler et al., 2010). Hence, we require all species to survive competitively and cooperative environment of stability in our prey-predator model for sustainability.

We will base our sustainability model on the work of May and Leonard in order to demonstrate these features for any dynamical systems model of sustainability (May and Leonard, 1975). The model has three sustainability indicators, which we denote by x, y and z. We assume that the time rate change in an indicator x

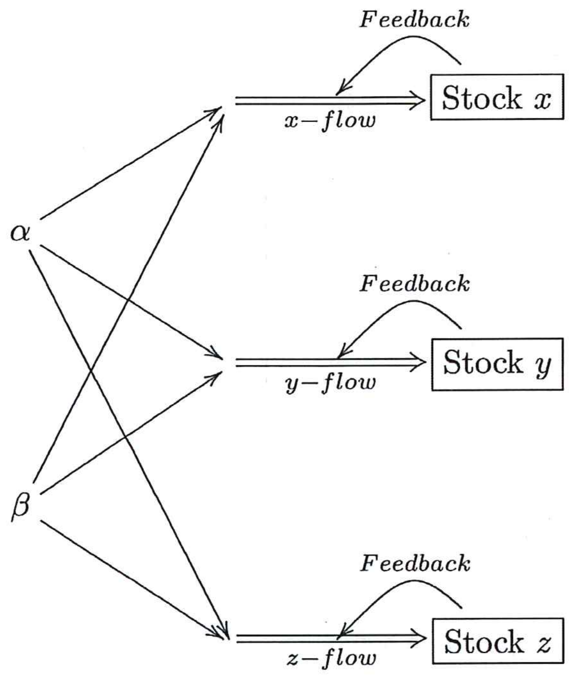

Fig. 5. Systems dynamic model with three sustainability indicators.

is proportional to its size, and that any two indicators can interact through a non-linear coupling term xy. Our model consists of three, first-order non-linear differential equations in terms of the sustainability indicators and the two parameters ¸ and ˇ.

$$\frac{dx}{dt} = x[1 - x - \text{¸}\, y - \text{ˇ}\, z]$$

$$\frac{dy}{dt} = y[1 - \text{ˇ}\, x - y - \text{¸}\, z]$$

$$\frac{dz}{dt} = z[1 - \text{¸}\, x - \text{ˇ}\, y - z]$$

In Fig. 5, the stocks represent the sustainability indicators as boxes and the right hand side of these set of equations represents the flows as double arrows.

In May and Leonard's analysis, there are eight possible fixed point equilibria that represent eight stable states. In a three-dimensional state space, these are: (0,0,0), three single solutions of the form (1,0,0), three solutions of the form (1 − ¸ ,1 − ˇ ,0)/(1 − ¸ ˇ) and the point (1,1,1)/(1 + ¸ +ˇ). However, we are interested in the dynamical systems whose solutions are limit cycles. First, the number of parameters can be reduced to two, where symmetry of the system has been exploited. Second, their analysis, and important here, shows that the parameters can be related by the limit cycle condition ¸ +ˇ =2. The parameters, ¸ and ˇ , divide the parameter space into three regions of stability. It is important to realize that these are partly the conditions for sustainability because we also require the system to be stable and resilient under the notion of "multiple equilibria". We suggest all "dynamical systems" are unsustainable unless the system entails internal stability and resilience components, such as adaptive bifurcations. While May and Leonard's paper does not address the issue of bifurcations or "multiple equilibria", it does provide a good starting point for such an investigation to include the resilience of the system.

However, the May and Leonard's model is not complete in its description of sustainability. While it determines equilibrium at fixed points and limit cycles with mathematical precision, sustainable systems are often far from equilibrium requiring additional approaches to modeling ecological stability. We merely point the inadequates of models only using a set of first order differential equations without considering them as complex ecosystems.

Now that we have described a general model for multiple equilibria, we will now define and discuss, as an application, a hypothetical model for nitrogen runoff into a lake and the environmental effects of cyclical precipitation (see Fig. 6). Runoff (x) is defined as a surface deposition of chemical fertilizer (nitrogen) from residential, agricultural and/or industrial land into the Lake. Rain (y) is defined as the hydrological cycle of surface water evaporation leading to cloud formation and precipitation. Lake (z) is defined as a self-contained body of surface water that is the variable recipient of Runoff and Rain. As in any dynamic system model, we simplify the nitrogen cycle (N) and hydrological cycle (H_2O) into the single sustainability indicator Runoff and Rain, respectively. In our simulations, we set the initial values for the amount of nitrogen runoff (x in red) to 1.0, the amount of rain (y in green) to 0.8 and the level of nitrogens in lake (z in blue) to 0.2. All simulations were produced with numerical data generated by the modeling program NetLogo (Wilensky, 1999).

To obtain stability, we set the values of the parameters so that the limit cycle condition is satisfied. Let ¸ =0.2 and ˇ =1.8. Then we observe a stable limit cycle behavior in the sustainability indicators as shown in Fig. 7. We note that any initial conditions can be given but exhibit limit cycle behavior only produces a sustainable system. Initially, rain (H_2O) occurs followed by a decreasing flow of nitrogen runoff (NH_4 and NO_3), and finally an increasing level of nitrogens in the lake. Since limit cycle behavior is quasi-periodic, reversing

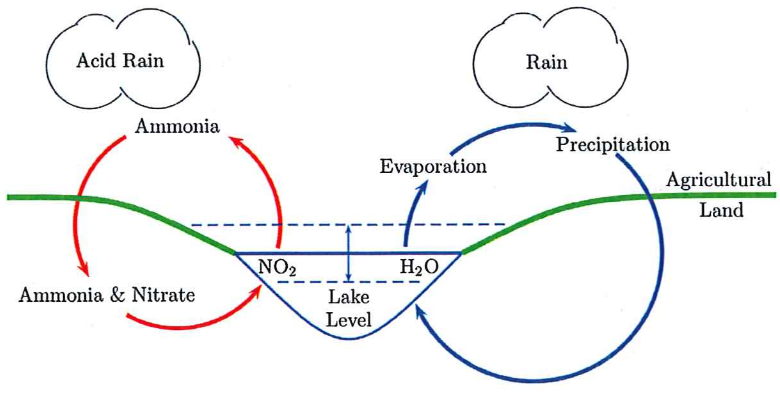

Fig. 6. Visual model of a lake with hydrological and nitrogen cycles. (For interpretation of the references to color in the text, the reader is referred to the web version of the article.)

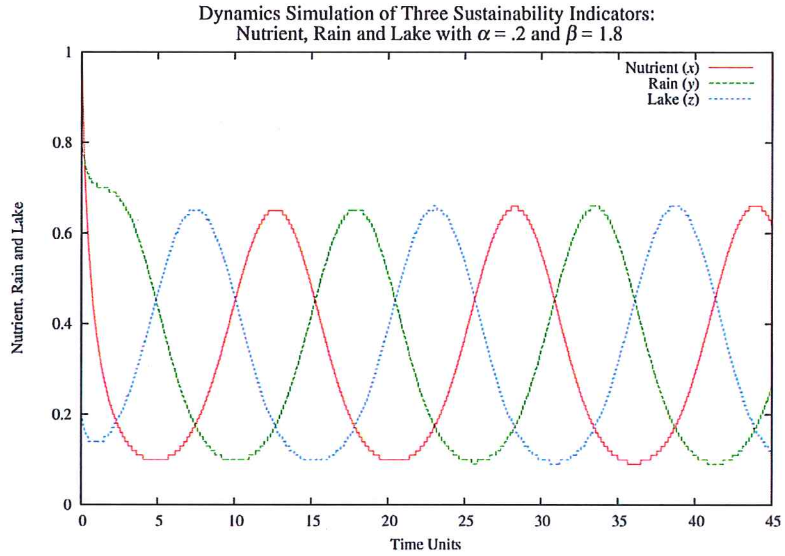

Fig. 7. The dynamics simulation of three sustainability indicators satisfying the limit cycle condition ¸ +ˇ =2 (¸ =.2,ˇ =1.8).

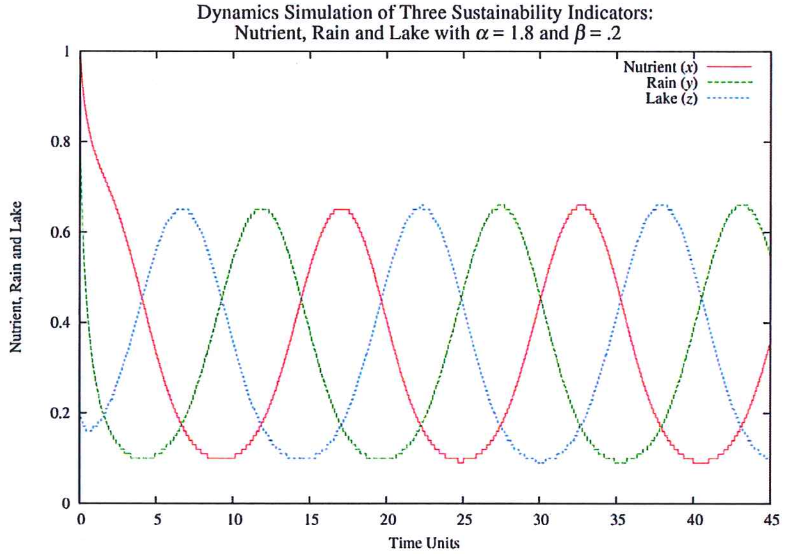

Fig. 8. The dynamics simulation of three sustainability indicators satisfying the limit cycle condition ¸ +ˇ =2 (¸ =1.8,ˇ =.2).

the values of the parameters allows one to shift one sustainability indicator relative to another. In Fig. 8, we set ¸ =1.8 and ˇ =0.2, where we observe the rain to subside sooner accompanied by a longer period of nitrogen runoff. We call this situation sustainable since none of the sustainability indicators vanish or become large un-measurable quantities.

In contrast, the system can become unstable or unsustainable when the limit cycle condition is violated. Unsustainable means that some or all of the sustainability indicators either become an equilibrium point or approach large, unmeasurable values. Any sustainability indicator or set of indicators that vanish is undesirable since the system becomes extinct, and, thus is unsustainable. In particular, set ¸ =2 and ˇ =1. Then Fig. 9 corresponds to the equilibrium point (1,0,0), where two of the three sustainability indicators vanish with a persistence level of nitrogens in the lake after a peak in the rain occurs. It is unrealistic to have vanishing sustainability indicators because one should always expect a certain level of an indicator in the environment. In Fig. 10, reversing the values of the parameters, we observe a peak in nitrogen runoff and vanishing rain with a constant level of water in the lake.

In Figs. 11 and 12, the only equilibrium point, namely (1,1,1)/(1 + ¸ +ˇ), has special importance since all the sustainability indicators approach the same, non-zero value of 0.53 for large times. Yet one might well expect, in general, that the values of sustainability indicators be distinct but this would not be an equilibrium point as shown by May and Leonard (1975). With these parameters, our sustainability model is similar to the Steady State Run of the Limits to Growth model but each of the parameters in this model approach different values. Our model only has three sustainability indicators while The Limits to Growth model have (5) variables. Whether using real world data or not, the two caveats for this model are: (1) we cannot accurately identify or predict the origin of point sources of any sustainability indicator or its quantity, and (2) we have neglected other possible causal factors in the environment. This sustainability model can, however, suggest general trends of the sustainability indicators (Jin et al., 1995). In the former caveat, a real world data assessment may include, nitro-

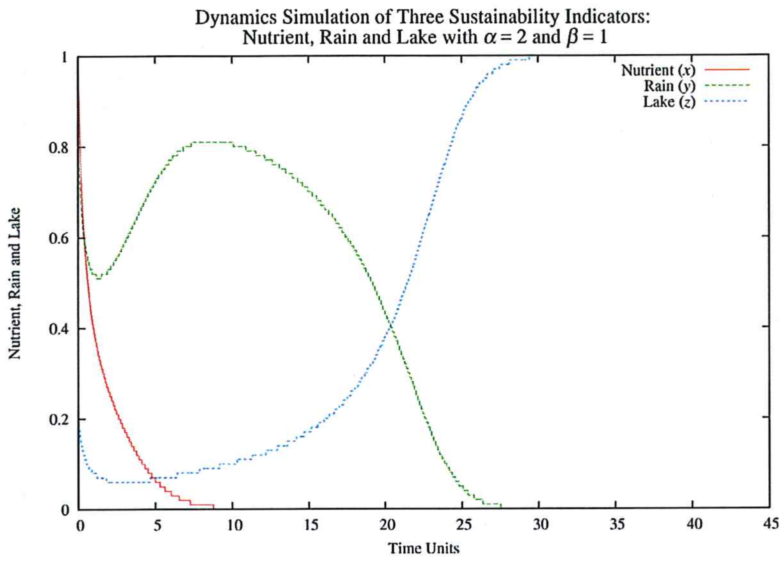

Fig. 9. The dynamics simulation of three sustainability indicators violating the limit cycle condition $\alpha+\beta>2$ ($\alpha=2, \beta=1$).

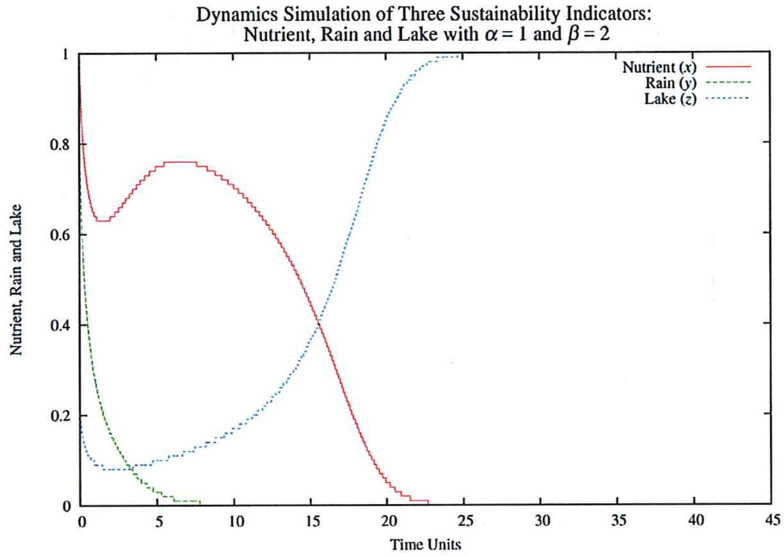

Fig. 10. The dynamics simulation of three sustainability indicators violating the limit cycle condition $\alpha+\beta>2$ ($\alpha=1, \beta=2$).

gen levels in the lake could be attributed to nearby homes and the home buying habits over the course of a year, which change from year to year as well as nitrogen runoff from these homes. In the later caveat, the increasing levels of nitrogen could arise from other factors, such as dead plant matter in the lake (detritus) or nitrification from acid rain. Also, decreasing levels of nitrogen could be due to other factors, such as denitrification from microorganisms in the soil and water. We remark that random disruptions to the dynamical system can be modeled as a stochastic differential equation, our equations with a noise term, but this is outside the scope of this paper. In spite of its current limitations this model can serve as a general guide to what constitutes a sustainable, manageable system in terms of the societal, economic and ecological concerns.

5. Future research

As with the Limits to Growth model, our goal is to simulate possible, as well as, probable realities. The Club of Rome employed statistical data to model future scenarios based on past information and projecting those patterns into “standard runs”. These runs, in turn, could be used to simulate different trajectories by manipulating the data. While some trajectories were more optimistic than the original standard run, all future scenarios suggested inevitable resource exhaustion due to population growth beyond carrying capacity and likely population extinction once resources are exhausted. Only the “steady state” model left room for long-term sustainability. Growth in a dynamical system is normal and necessary, but unlimited growth in a finite system can be disastrous. The multiple equilibria sustainability model takes into consideration the statistical limitations in growth models and simulates probable present-to-future dynamical behavior using polycentric non-linear equations. We have seen that “limit cycles” provide states with both competitive and cooperative behaviors – the features for any dynamical system in modeling sustainability. Certain mathematical relationships between the parameters ($\alpha+\beta=2$) provide the conditions for the presence of limit cycles. One possible way to extend these types of models is to require

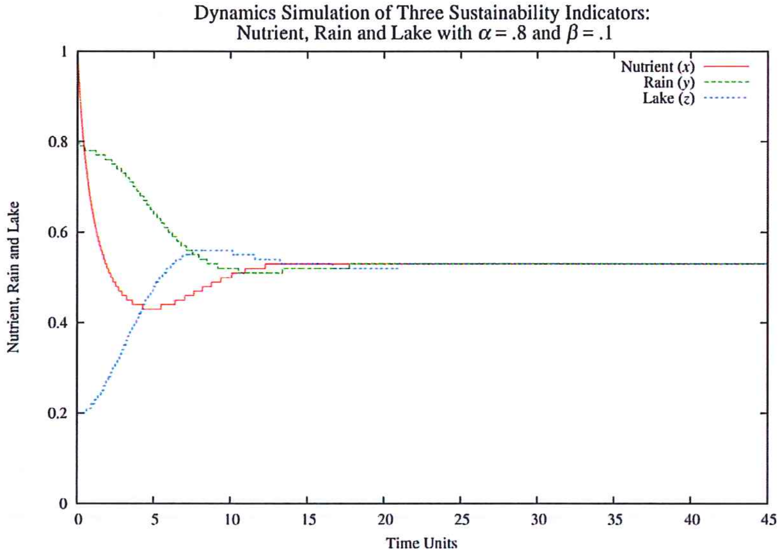

Fig. 11. The dynamics simulation of three sustainability indicators violating the limit cycle condition ¸ +ˇ <2 (¸ =.8,ˇ =.1).

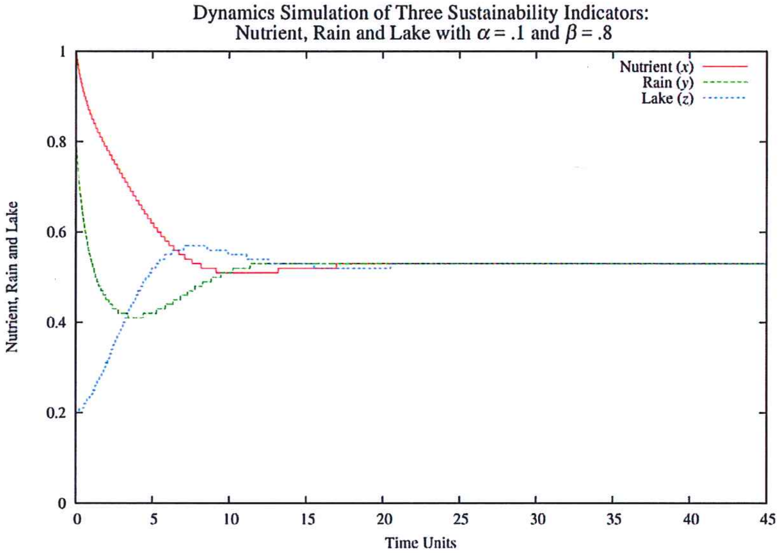

Fig. 12. The dynamics simulation of three sustainability indicators violating the limit cycle condition ¸ +ˇ <2 (¸ =.1,ˇ =.8).

the parameters to be functions of time so that the "limit cycle" condition is met, namely ¸ (t)+ˇ (t)=2.

The coherence, validity and trustworthiness of this sustainability model are important for its adoption, but it must fit into real world scenarios. So, further study would be needed to test the strength of the model through empirical research using measurable variables such as local rainfall, fertilizer volume, nitrogen loading, speed of nitrogen movement across impervious surfaces, nitrate concentration, and urban green space distribution to simulate spatial and temporal fertilizer runoff rates into a lake or bay (Laband, 2005; Tsihrintzis et al., 1996). The sustainability model can be expanded to include more sustainability indicators (variables), so a comparison of the model's results to real data is feasible, and even the predictions of the Limits to Growth model. The collection and manipulation of physical, economic, societal, biological, and chemical data can help develop a more reliable "sustainability modeling" tool in the future to map "multiple equilibria." Moreover, our approach can be expanded to a set of stochastic differential equations to include random disruption of the system. If proven to be reliable in its predictions sustainability modeling has significant environmental policy and systems management implications at local and regional levels, and possibly, be aggregated to a global scale.

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